Lecture 5

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Coherent states

As we have seen and you will still proove number states do not have a well defined electric field. Thus, number states are not suitable to describe the quantized version of a classical field $\boldsymbol{E}(\boldsymbol{r}, t)$. Can we find another quantum state that corresponds to $\boldsymbol{E}(\boldsymbol{r}, t)$ in the classical limit? The intuitive way to achieve that would be a state with a huge number of photons $N \gg 1$. However, as we have shown, the fact that $\langle n|\hat{E}(\boldsymbol{r}, t)|n\rangle = 0$ does not depend on the photon number. Instead, since $\hat{E}$ depends on a sum of $\hat{a}$ and $\hat{a}^\dagger$, we could construct a state that consists of Fock states with numbers $|n\rangle$ and $|n+1\rangle$.

Definition: Coherent states $|\alpha\rangle$ are eigenfunctions of the annihilation operator $\hat{a}$, so that:

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{1}$$

Because Fock states are a complete ONS, we can expand a coherent state in terms of number states, so that:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle\,. \tag{2}$$

If we act with the annihilation operator on this expansion, we get:

$$\hat{a}\,|\alpha\rangle = \sum_{n=0}^{\infty} c_n \hat{a}\,|n\rangle = \sum_{n=1}^{\infty} \sqrt{n} c_n\,|n-1\rangle\,. \tag{3}$$

Because $|\alpha\rangle$ is an eigenfunction of $\hat{a}$ we find the condition:

$$\sum_{n=1}^{\infty} \sqrt{n} c_n\,|n-1\rangle = \alpha \sum_{n=0}^{\infty} c_n\,|n\rangle = \alpha \sum_{n=1}^{\infty} c_{n-1}\,|n-1\rangle\,. \tag{4}$$

By comparison of the coefficients of the same Fock state, we find the recurrence relation:

$$\sqrt{n} c_n = \alpha c_{n-1}, \tag{5}$$

which we can rewrite:

$$c_n = c_0 \frac{\alpha^n}{\sqrt{n!}} \tag{6}$$

The only free parameter is the first coefficient c_0, which can be fixed by normalization:

$$\langle\alpha|\alpha\rangle = |c_0|^2 \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} \frac{(\alpha^*)^n \alpha^{n'}}{\sqrt{n!n'!}}\,\langle n|n'\rangle = |c_0|^2 \sum_{n=0}^{\infty} \frac{(|\alpha|^2)^n}{n!} = |c_0|^2 e^{|\alpha|^2} \stackrel{!}{=} 1 \tag{7}$$

and therefore $c_0 = e^{-\frac{|\alpha|^2}{2}}$. Thus, a coherent state is constructed by:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n (\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle \tag{8}$$

No let us show, that any state can be expanded in coherent states. Therefore we write:

$$\begin{aligned}
\int \mathrm{d}\alpha \, |\alpha\rangle \langle\alpha| &= \sum_{n,m} \int \mathrm{d}\alpha \, |n\rangle \langle n|\alpha\rangle \langle\alpha|m\rangle \langle m| \\
&= \sum_{n,m} \frac{|n\rangle \langle m|}{\sqrt{n!m!}} \underbrace{\int \mathrm{d}\alpha \, \mathrm{e}^{-|\alpha|^2} \alpha^n \alpha^{*m}}_{=\pi n! \delta_{nm}} \\
&= \pi \sum_n |n\rangle \langle n|
\end{aligned}$$

Therefore we have found that the unity operator in terms of coherent states is:

$$\frac{1}{\pi} \int \mathrm{d}\alpha \, |\alpha\rangle \langle\alpha| = \hat{I} \tag{9}$$

The electric field of a coherent state

The aim of the whole operation above was to construct a state with the proper classical limit. So let us calculate the expectation value of the electric field operator:

$$\begin{aligned}
\langle\alpha|\hat{E}(\boldsymbol{r},t)|\alpha\rangle &= \mathrm{i}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \, \langle\alpha|\hat{a} \, \mathrm{e}^{\mathrm{i}(\boldsymbol{r}\cdot\boldsymbol{k}-\omega t)} - \hat{a}^\dagger \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{r}\cdot\boldsymbol{k}-\omega t)}|\alpha\rangle \\
&= \mathrm{i}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left(\alpha \, \mathrm{e}^{\mathrm{i}(\boldsymbol{r}\cdot\boldsymbol{k}-\omega t)} - \alpha^* \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{r}\cdot\boldsymbol{k}-\omega t)}\right) \\
&= \mathrm{i}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} |\alpha| \left(\mathrm{e}^{\mathrm{i}(\boldsymbol{r}\cdot\boldsymbol{k}-\omega t+\phi)} - \mathrm{e}^{-\mathrm{i}(\boldsymbol{r}\cdot\boldsymbol{k}-\omega t+\phi)}\right) \\
&= -2\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} |\alpha| \sin(\boldsymbol{r}\cdot\boldsymbol{k}-\omega t+\phi),
\end{aligned} \tag{10}$$

which corresponds to the classical electrical field. In the exercises you will show, that the variance of the electric field is

$$\langle\alpha|\Delta\hat{E}^2(\boldsymbol{r},t)|\alpha\rangle = \frac{\hbar\omega}{2\varepsilon_0 V}. \tag{11}$$

This corresponds to the variance of the vacuum and thus shows the definition of coherent states as states with minimal uncertainty.

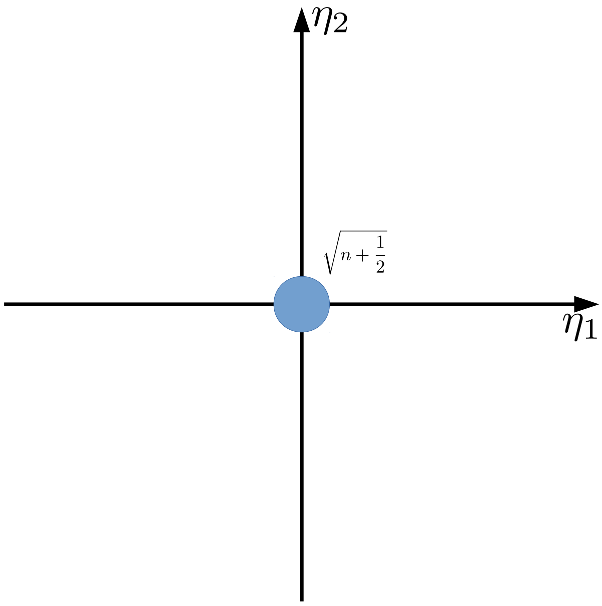

Figure 1: A Fock or number state in phase space corresponds to a circular probability distribution around zero. The variance has a radius of $\sqrt{n+\frac{1}{2}}$.

Photon statistics of coherent states

Since we can expand coherent states into number states, it is a straightforward question to ask, what the expectation value of the photon number is in such a state. As you will show it corresponds to:

$$\langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2 \equiv \overline{n} \tag{12}$$

We can use this relation to find an expression for the probability distribution of the photon number. The probability to find n photons in a coherent state $|\alpha\rangle$ can be calculated as follows:

$$\begin{aligned} P_n = |\langle n|\alpha\rangle|^2 &= \left| \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{m=0} \frac{\alpha^m}{\sqrt{m!}} \langle n|m\rangle \right|^2 \\ &= \mathrm{e}^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \\ &= \mathrm{e}^{-\overline{n}} \frac{\overline{n}^n}{n!} \end{aligned}$$

This corresponds to a Poisson-distribution!

Fock and coherent states in phase space

As we know from classical mechanics, each configuration at point $\boldsymbol{r}$ with momentum $\boldsymbol{p}$ can be represented by a point $(\boldsymbol{r}, \boldsymbol{p})$ in phase space. We can perform a similar consideration in quantum optics. Let

$$\boldsymbol{r} \to \hat{\eta}_1 = \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right)$$
$$\boldsymbol{p} \to \hat{\eta}_2 = \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right)$$

From these conjugated operators $\hat{\eta}_1$ and $\hat{\eta}_2$ we can build a 2D-phase space associated with a single-mode field. The expectation values of both operators with respect to a Fock state are

$$\langle n|\hat{\eta}_{1/2}|n\rangle = 0.$$

The variance, however, is non-vanishing:

$$\langle n|\Delta\hat{\eta}_{1/2}^2|n\rangle = \frac{1}{4}(2n+1).$$

Since $\hat{\eta}_1$ and $\hat{\eta}_2$ do not commute we find the incertainty:

$$\left\langle \Delta\hat{\eta}_1^2 \right\rangle \left\langle \Delta\hat{\eta}_2^2 \right\rangle \geq \frac{1}{16}$$

Therefore we find for a number state in phase space the following relation:

$$\left\langle \Delta\hat{\eta}_1^2 \right\rangle + \left\langle \Delta\hat{\eta}_2^2 \right\rangle = n + \frac{1}{2} \tag{13}$$

which describes a circle with radius $\sqrt{n+\frac{1}{2}}$, as depicted in Fig. 1.

The description of a coherent state in phase space requires a bit more work. For the expectation values we find the interesting relation:

$$\begin{aligned} \langle\alpha|\hat{\eta}_1|\alpha\rangle &= \frac{1}{2}(\alpha+\alpha^*) = \mathrm{Re}(\alpha) \\ \langle\alpha|\hat{\eta}_2|\alpha\rangle &= \frac{1}{2i}(\alpha-\alpha^*) = \mathrm{Im}(\alpha) \end{aligned} \tag{14}$$

This means that we can map the complex α-space to the phase space of a coherent state. Let us, therefore, use the polar representation of α:

$$\alpha = |\alpha|\,\mathrm{e}^{\mathrm{i}\theta}$$

The uncertainty of the radial distance $|\alpha|$ corresponds to the standard deviation of the photon number $\sqrt{\bar{n}}$. Because $\hat{\eta}_1$ and $\hat{\eta}_2$, again, do not commute, there is also a phase uncertainty. The variance of both operators is given by

$$\left\langle \Delta\hat{\eta}_{1/2}^2 \right\rangle = \frac{1}{4}, \tag{15}$$

which is the same as the vacuum uncertainty, which again shows that (i) coherent states are states of minimal uncertainty and (ii) as seen in Fig. 2 coherent states can be viewed

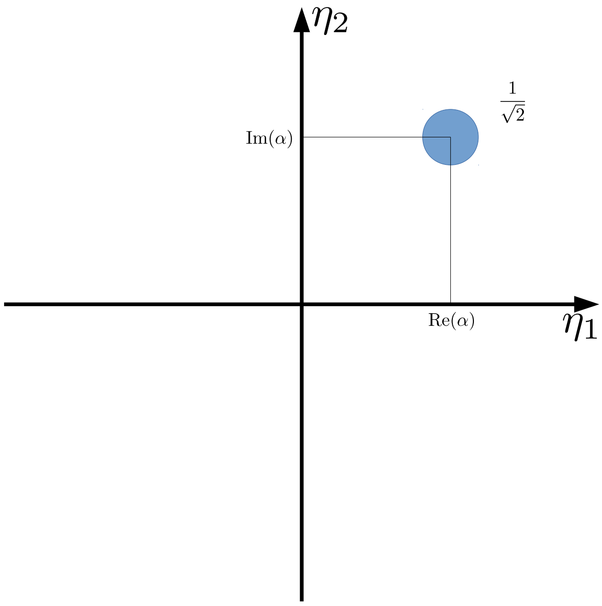

Figure 2: A coherent state $|\alpha\rangle$ in phase space has a circular probability distribution. The center of this distribution is at $(\mathrm{Re}(\alpha), \mathrm{Im}(\alpha)$. The $1-\sigma$-radius is the same as for the vacuum $\frac{1}{\sqrt{2}}$.

as vacuum states displaced in phase space. Indeed we can construct an operator $\hat{D}(\alpha)$ that, acted on the vacuum state, performs exactly that shift, i.e.:

$$\hat{D}(\alpha)\,|0\rangle = |\alpha\rangle \tag{16}$$

From Eq. (8) one might think that $\hat{D}(\alpha) = \exp(-|\alpha|^2/2)\exp(\alpha\hat{a}^\dagger)$. This, however, is not true, because this exponential operator would not be unitary. Instead we define:

$$\hat{D}(\alpha) = \mathrm{e}^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \tag{17}$$

With the Baker-Campbell-Hausdorff formula

$$\mathrm{e}^{\hat{A}+\hat{B}} = \mathrm{e}^{\frac{1}{2}[\hat{A},\hat{B}]}\,\mathrm{e}^{\hat{B}}\,\mathrm{e}^{\hat{A}} \tag{18}$$

and $\hat{A} = \alpha^*\hat{a}$ and $\hat{B} = \alpha\hat{a}^\dagger$ we see that this corresponds to

$$\hat{D}(\alpha) = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\,\mathrm{e}^{\alpha\hat{a}^\dagger}\,\mathrm{e}^{\alpha^*\hat{a}}, \tag{19}$$

where we note that the exponent of the annihilation operator acted on the vacuum state has no effect and, thus, we retain Eq. (8) and, thus, have obtaines the unitary shifting operator we were looking for.

Squeezed states

In the previous section both phase-space variables of a coherent state had the same variance:

$$\left\langle \Delta\hat{\eta}_{1/2}^2 \right\rangle = \frac{1}{4}. \tag{20}$$

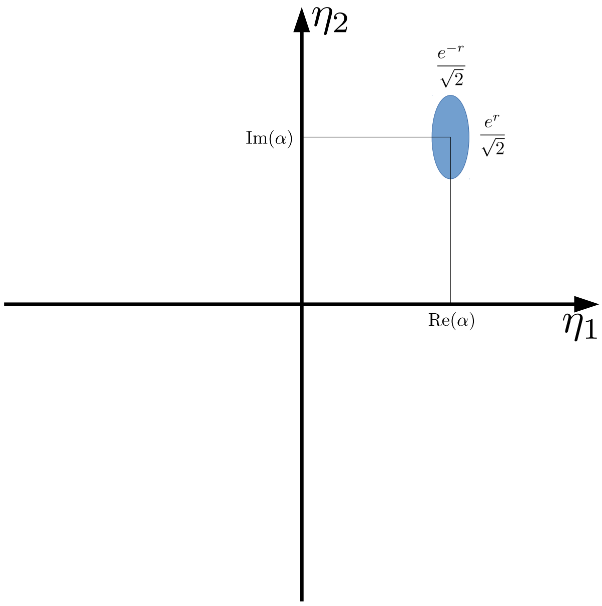

Figure 3: A squeezed state $|\alpha, \xi\rangle$ in phase space has an asymmetric probability distribution. The center of this distribution is at $(\mathrm{Re}(\alpha), \mathrm{Im}(\alpha)$. The $1-\sigma$-distance from the center is modified by a factor of $\mathrm{e}^{-|\xi|}$ or $\mathrm{e}^{|\xi|}$, respectively.

For various applications, however, it may be useful to construct states that have an uncertainty smaller than $\frac{1}{4}$ with respect to one obervable, while it is not important if another observable is defined less sharply. Therefore we aim to construct states, where:

$$\begin{aligned} \left\langle \Delta\hat{\eta}_1^2 \right\rangle &= \gamma \frac{1}{4} \\ \left\langle \Delta\hat{\eta}_2^2 \right\rangle &= \frac{1}{4\gamma}, \end{aligned} \tag{21}$$

where $\gamma \leq 1$. So that the product of both still satisfies $\langle \Delta\hat{\eta}_1^2 \rangle \langle \Delta\hat{\eta}_2^2 \rangle \geq \frac{1}{16}$. With a complex number ξ we can define the squeezing operator:

$$\hat{S}(\xi) = \mathrm{e}^{\frac{1}{2}\left[\xi^* \hat{a}^2 - \xi\left(\hat{a}^\dagger\right)^2\right]}, \tag{22}$$

which is a nonlinear operator in $\hat{a}$ and $\hat{a}^\dagger$. For consistency this operator has to be unitary:

$$\begin{aligned} \hat{S}^\dagger(\xi) &= \hat{S}(-\xi) \\ \hat{S}^\dagger(\xi)\hat{S}(\xi) &= \hat{S}(\xi)\hat{S}^\dagger(\xi) = \hat{I}. \end{aligned}$$

Since we have seen that a coherent state is just a displaced vacuum state, we can for simplicity study the action of this operator on the quantum vacuum to generate a squeezed vacuum state:

$$|\xi\rangle = \hat{S}(\xi)\,|0\rangle \tag{23}$$

Now let us proof that those states are, indeed, squeezed states. Therefore let us calculate the expectation values:

$$
\begin{aligned}
\langle\xi|\hat{\eta}_1|\xi\rangle &= \frac{1}{2}\,\langle 0|\hat{S}^\dagger(\xi)(\hat{a}+\hat{a}^\dagger)\hat{S}(\xi)|0\rangle \\
\langle\xi|\hat{\eta}_2|\xi\rangle &= \frac{1}{2i}\,\langle 0|\hat{S}^\dagger(\xi)(\hat{a}-\hat{a}^\dagger)\hat{S}(\xi)|0\rangle
\end{aligned} \tag{24}
$$

This, however, leaves us with the problem to calculate the operators $\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi)$ and $\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi)$. In order to do this we can use the decomposition of exponential operators:

$$
\mathrm{e}^{\hat{A}}\hat{B}\,\mathrm{e}^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}\left[\hat{A},[\hat{A},\hat{B}]\right] + \frac{1}{3!}\left[\hat{A}\left[\hat{A},[\hat{A},\hat{B}]\right]\right] + \ldots \tag{25}
$$

With $\hat{B}=\hat{a}$ or $\hat{B}=\hat{a}^\dagger$, respectively and $\hat{A}=\frac{1}{2}\left[\xi^*\hat{a}^2-\xi\left(\hat{a}^\dagger\right)^2\right]$ we find:

$$
\begin{aligned}
\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) &= \hat{a}+\xi\hat{a}^\dagger \\
\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi) &= \hat{a}^\dagger+\xi^*\hat{a}.
\end{aligned}
$$

Therefore:

$$
\langle\xi|\hat{\eta}_{1/2}|\xi\rangle = 0 \tag{26}
$$

and, following analogue considerations:

$$
\begin{aligned}
\langle\xi|\Delta\hat{\eta}_1^2|\xi\rangle &= \frac{1}{4}\,\mathrm{e}^{-2r} \\
\langle\xi|\Delta\hat{\eta}_2^2|\xi\rangle &= \frac{1}{4}\,\mathrm{e}^{2r},
\end{aligned} \tag{27}
$$

where we have assumed that $\arg(\xi)=0$ and r is the so-called *squeezing parameter*. Finally we need to mention, that, of course, not only the vacuum can be squeezed, but any coherent state by applying the squeezing operator:

$$
|\alpha,\xi\rangle = \hat{S}(\xi)\,|\alpha\rangle \tag{28}
$$

Such a state, that is squeezed in η_1-direction is shown in Fig. 3.