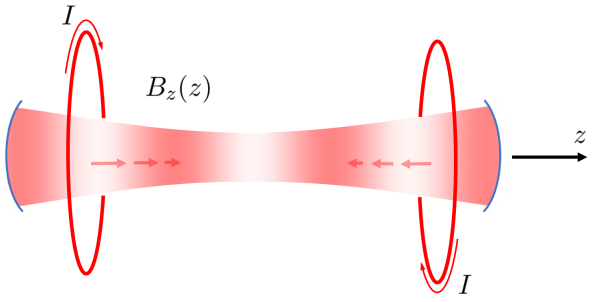
Bose-Einstein Condensates

1 Creating BECs by Evaporative Cooling

In the previous lecture we have learned, that the kinetic energy of atoms in a magneto-optical trap (MOT) can be reduced by the absorption of a photon, which as an frequencies, that is slightly lower than the frequency, needed to stimulate a transition in the atom. However, this method of *laser cooling* is only capable to reduce the kinetic energy up to ∼µK (relation $E_{kin} \sim k_B\, T$). For further cooling, therefore, an additional method is used, the *evaporative cooling*. Here, we want to discuss this method with the help of a simple one-dimensional picture :

a)

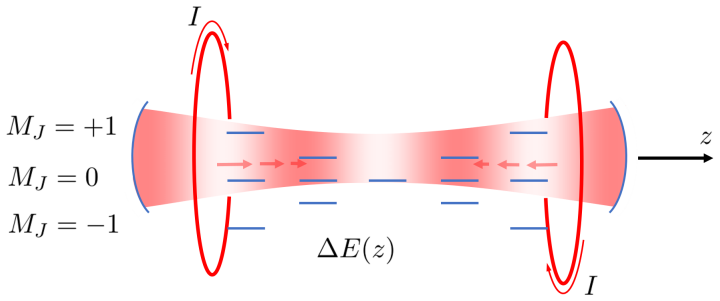


The method of evaporative cooling makes use of the position dependent magnetic field of the MOT, which is produced by a pair of anti-Helmholtz coils. The resulting magnetic field along the z axis is

$$B_z(z) = -2B_0' z \tag{1}$$

Unlike the laser cooling process, which works du to internal processes *in* the atom, i.e. transitions of the electron, the process of evaporate cooling can be understood by treating the entire atom as a single particle.

b)



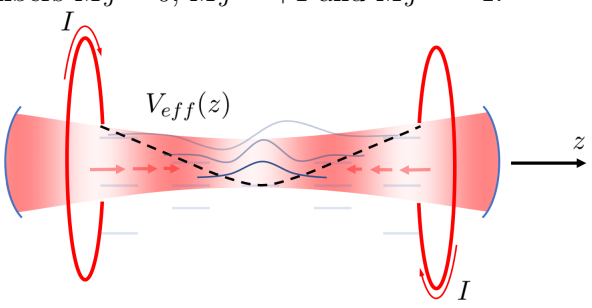
The atoms in the MOT are affected by the magnetic field.

Since the atoms are composite particles build up from Fermions (positrons, neutrons, electrons), a special species of atoms can have either a integer or a half-integer total spin J. This also means, that the atoms behave as Bosons or Fermions, which will be important later on. As an example we look at Hydrogen which consists of an proton and a single electron. Due to that, the hydrogen atom can have a total spin of $J = 0$ or $J = 1$. In the latter case, the external magnetic field would couple to the total spin of the atom and generates a shift of energy of the atoms' center of mass motion (in this simple model, we neglect the effect on internal degrees of freedom!). In the local frame of the atom, where the z-axis is defined along the magnetic field vector, this energy shift reads

$$\Delta E(z) = g_J \, \mu_B |B_z(z)| M_J \, . \tag{2}$$

With the Landé factor g_J, and the Bohr magnetron $\mu_B = e\hbar/2m$. For hydrogen with $J = 1$, this leads to an position dependent Zeeman splitting for the three possible values of magnetic quantum numbers $M_J = 0$, $M_J = +1$ and $M_J = -1$.

c)

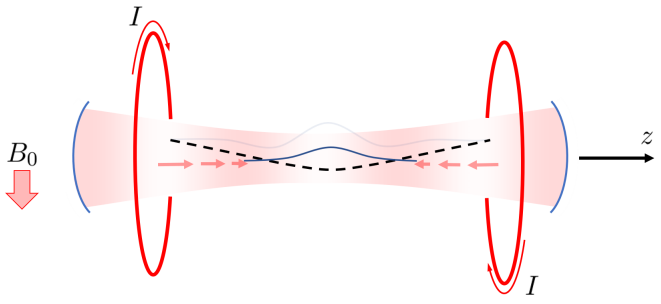


The position dependent energy correction can be seen as an effective potential for the atoms motion.

$$V_{eff}(z) = (g_J M_J) \frac{e\hbar B_0'}{m} \, |z| \tag{3}$$

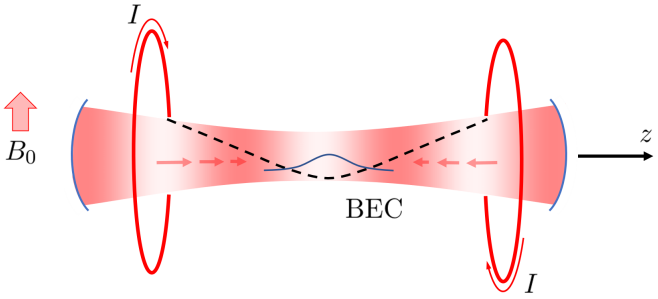
This potential is attractive in the case of $M_J > 0$. Due to the rest temperature after laser cooling ($\sim$ µK), all states below the corresponding $k_B \, T$ are populated. Depending on the total spin of the atoms, the population of the states satisfies a bosonic or fermionic particle statistics. That means, the states can be populated by only two particles per state (fermionic case) or by several atoms (bononic case). We want to concentrate on the bononic case, where, due to particle statistics, most of the atoms are already in the ground state.

d)



In the next step, we want to get rid of all states but the ground state. This can be done by decreasing the intensity of the laser light (or turn off the laser completely) and reduces the magnetic field strength. As a consequence the potential is flattened. Therefore, higher energy states become unbound, such that the particles in this state *evaporate* from the system. If particles with higher energy leave the system, this results in a lower temperature.

e)



In order to store all remaining particles, we can increase the intensity of the laser and the magnetic field strength again. All atoms are now *condensed* in the ground state. In result, we have created a *Bose-Einstein condensate* (BEC).
Since a Bose-Einstein condensate consists of a huge number of particles in the same quantum state, the overlay of quantum mechanical probability densities for every single particle $N|\psi_i(r)|^2$ becomes equal to the classical distribution of atoms $n(\vec{r})$ of the whole system, that can analyzed on mesoscopic scales. However, it retains all its quantum properties.

2 Theoretical Description of a BEC

We have learned, that a BEC is a collective phenomenon, that is described by a lot of identical quantum states. Therefore, it is natural to start at the level of a many particle theory and built a Fock space from single particle wave functions. In our special case of a BEC, all single particle wave functions describe the same state, i.e. $\psi_i = \psi(\vec{r}_i)$. Therefore, the many particle wave function of the system can be written as

$$|\Psi\rangle = \sqrt{N} \prod_{i=1}^{N} \int \mathrm{d}\vec{r}_i^{\,3} \psi(\vec{r}_i) \, |\vec{r}_i\rangle \, . \tag{4}$$

This N-particle wave function is normalizes with respect to the integral over N copies of the configuration space

$$\begin{aligned}
\langle\Psi|\Psi\rangle &= N\prod_{i=1}^{N}\int \mathrm{d}\vec{r}_i^{\,3}\int \mathrm{d}\vec{r}_i\,'^{\,3}\psi^\dagger(\vec{r}_i\,')\psi(\vec{r}_i)\,\langle\vec{r}_i\,'|\vec{r}_i\rangle \\
&= N\prod_{i=1}^{N}\int \mathrm{d}\vec{r}_i^{\,3}\int \mathrm{d}\vec{r}_i\,'^{\,3}\psi^\dagger(\vec{r}_i\,')\psi(\vec{r}_i)\,\delta(\vec{r}_i\,'-\vec{r}_i) \qquad (5)\\
&= N\prod_{i=1}^{N}\underbrace{\int \mathrm{d}\vec{r}_i^{\,3}|\psi(\vec{r}_i)|^2}_{=1} \\
&= N\,.
\end{aligned}$$

The dynamics of the system is governed by the Hamiltonian

$$H=\sum_{i=1}^{N}H_i+\sum_{\substack{i,j=1\\ j<i}}^{N}W_{ij}\,, \qquad (6)$$

which is constructed from the single particle Hamiltonians H_i

$$H_i=-\frac{\hbar^2}{2m}\,\Delta_{\vec{r}_i}+V(\vec{r}_i) \qquad (7)$$

and an additional interaction W_{ij} between the particles, which we want to be

$$W_{ij}=W_0\,\delta(\vec{r}_i-\vec{r}_j)\,. \qquad (8)$$

This means, that a particle with index i feels an infinitely strong repulsive potential at the position of the particle with index j. Therefore, two particles are not able to be at the same position $\vec{r}_i\neq\vec{r}_j$. Thanks to the equality of all singe particle wave equations, this interaction between two particles in the BEC can be rewritten in the form of a *self interaction* of every single particle. In order to show this, we straight forwardly calculate

$$\langle\Psi|H|\Psi\rangle-E\langle\Psi|\Psi\rangle = \sum_{i=1}^{N}\langle\Psi|H_i|\Psi\rangle+\sum_{\substack{i,j=1\\ j<i}}^{N}\langle\Psi|W_{ij}|\Psi\rangle-E\langle\Psi|\Psi\rangle=0\,. \qquad (9)$$

Starting with the sum over the single particle Hamiltonians

$$\begin{aligned}
\sum_{i=1}^{N}\langle\Psi|H_i|\Psi\rangle &= N\sum_{i=1}^{N}\int \mathrm{d}\vec{r}_i^{\,3}\psi^\dagger(\vec{r}_i)H_i\psi(\vec{r}_i)\prod_{\substack{k=1\\ k\neq i}}^{N}\underbrace{\int \mathrm{d}\vec{r}_k^{\,3}|\psi(\vec{r}_k)|^2}_{=1} \qquad (10)\\
&= \underbrace{N\sum_{i=1}^{N}\int \mathrm{d}\vec{r}^{\,3}\psi^\dagger(\vec{r})\left(-\frac{\hbar^2}{2m}\,\Delta_{\vec{r}}+V(\vec{r})\right)\psi(\vec{r})}_{N-\text{times the same integral}} \\
&= N^2\int \mathrm{d}\vec{r}^{\,3}\psi^\dagger(\vec{r})\left(-\frac{\hbar^2}{2m}\,\Delta_{\vec{r}}+V(\vec{r})\right)\psi(\vec{r})\,,
\end{aligned}$$

in the same way we can rewrite the interaction

$$\begin{aligned}\sum_{\substack{i,j=1\\ j<i}}^{N}\langle\Psi|W_{ij}|\Psi\rangle &= N\sum_{\substack{i,j=1\\ j<i}}^{N}\int\mathrm{d}\vec{r}_i^{\,3}\int\mathrm{d}\vec{r}_j^{\,3}|\psi(\vec{r}_i)|^2|\psi(\vec{r}_j)|^2W_0\,\delta(\vec{r}_i-\vec{r}_j)\prod_{\substack{k=1\\ k\neq i,j}}^{N}\underbrace{\int\mathrm{d}\vec{r}_k^{\,3}|\psi(\vec{r}_k)|^2}_{=1}\\ &= \underbrace{N\sum_{\substack{i,j=1\\ j<i}}^{N}\int\mathrm{d}\vec{r}^{\,3}\,W_0\,|\psi(\vec{r})|^4}_{N(N-1)/2-\text{times}} \qquad (11)\\ &= N^2\int\mathrm{d}\vec{r}^{\,3}\psi^\dagger\left(\frac{N-1}{2}\,W_0\,|\psi(\vec{r})|^2\right)\psi(\vec{r})\;.\end{aligned}$$

Moreover, we want the energy to be $E = N\mu$, where we introduce the *chemical potential* μ, what gives

$$E\langle\Psi|\Psi\rangle = NE = N^2\mu = N^2\mu\underbrace{\int\mathrm{d}\vec{r}^{\,3}|\psi(\vec{r})|^2}_{=1} = N^2\int\mathrm{d}\vec{r}^{\,3}\psi^\dagger(\vec{r})\;\mu\;\psi(\vec{r})\;. \qquad (12)$$

Putting the three parts together, we finally obtain

$$\int\mathrm{d}\vec{r}^{\,3}\psi^\dagger(\vec{r})\left(-\frac{\hbar^2}{2m}\Delta_{\vec{r}}+V(\vec{r})+\frac{N-1}{2}\,W_0|\psi(\vec{r})|^2\psi(\vec{r})-\mu\right)\psi(\vec{r})=0\,. \qquad (13)$$

Since every single particle in the condensate now has the same wave function, we can introduce a new wave function $\psi_{\mathrm{BE}} = \sqrt{N}\psi$ which describes the whole BEC as one object, instead of a Fock state in a many particle theory. For large $N \gg 1$ we get

$$\int\mathrm{d}\vec{r}^{\,3}\psi^\dagger_{\mathrm{BE}}(\vec{r})\left(-\frac{\hbar^2}{2m}\Delta_{\vec{r}}+V(\vec{r})+\frac{1}{2}\,W_0|\psi_{\mathrm{BE}}(\vec{r})|^2-\mu\right)\psi_{\mathrm{BE}}(\vec{r})=0\,, \qquad (14)$$

which after variation $\psi_{\mathrm{BE}} + \delta\psi_{\mathrm{BE}}$ leads to the *Gross-Pitaevskii equation* (1961) for the BEC:

$$\left(-\frac{\hbar^2}{2m}\Delta_{\vec{r}}+V(\vec{r})+\;W_0|\psi_{\mathrm{BE}}(\vec{r})|^2-\mu\right)\psi_{\mathrm{BE}}(\vec{r})=0\,. \qquad (15)$$

Therefore, the dynamics of a single particle in the BEC is described by a non-linear Schrödinger equation. With the normalization condition

$$\langle\psi_{\mathrm{BE}}|\psi_{\mathrm{BE}}\rangle = \int\mathrm{d}\vec{r}^{\,3}|\psi_{\mathrm{BE}}(\vec{r})|^2 = \int\mathrm{d}\vec{r}^{\,3}n(\vec{r}) = N \qquad (16)$$

we introduce the number density $n(\vec{r}) = |\psi_{\mathrm{BE}}(\vec{r})|^2$ of the BEC.

3 Thomas-Fermi Approximation

In the Thomas-Fermi approximation, we neglect the kinetic energy of the BEC, which is considered to be small in comparison with the external potential and the self-interaction of the BEC:

$$\left(\cancel{-\frac{\hbar^2}{2m}\Delta_{\vec{r}}} + V(\vec{r}) + W_0 n(\vec{r}) - \mu\right)\psi_{\mathrm{BE}}(\vec{r}) = 0 \tag{17}$$

In this case, the Gross-Pitaevskii equation becomes an algebraic equation, which we can solve for the number density $n(\vec{r})$:

$$n(\vec{r}) = \frac{\mu - V(\vec{r})}{W_0} \tag{18}$$

Here, we want to discuss the case of an one-dimensional oscillator potential $V(x) = \frac{1}{2}m\omega x^2$. The number density then reads

$$n(x) = \frac{\mu - \frac{1}{2}m\omega x^2}{W_0}\,. \tag{19}$$

With the help of the normalization condition, we get

$$\int_{-x_0}^{x_0} \mathrm{d}x\; n(x) = \frac{2\mu x_0}{W_0} - \frac{m\omega x_0^3}{3W_0} \stackrel{!}{=} N\,, \tag{20}$$

which leads to an expression for the chemical potential:

$$\mu(x_0) = \frac{W_0 N}{2x_0} + \frac{1}{6}m x_0^2 \omega\,. \tag{21}$$

Since the BEC consists of particles in the ground state, we have to minimize $\mu(x_0)$, in order to find the edges of the BEC

$$\frac{\partial\mu(x_0)}{\partial x_0} \stackrel{!}{=} 0 \quad \Rightarrow \quad x_0 = \left(\frac{3W_0N}{2m\omega}\right)^{1/3} \quad , \quad \mu = \frac{1}{2}\,m\omega\,\left(\frac{3W_0N}{2m\omega}\right)^{2/3}\,. \tag{22}$$

The Bose-Einstein condensate in Thomas-Fermi approximation has the number density

$$n(x) = \frac{1}{2}\frac{m\omega}{W_0}\left[\left(\frac{3W_0N}{2m\omega}\right)^{2/3} - \; x^2\right]\,. \tag{23}$$

This first guess of the BEC's shape is often used as a starting point for numerical investigations of the full non-linear Gross-Pitaevskii equation.