Lecture 2: Born approximation, Feynman diagrams

In the previous lecture we have found the formal solution of the stationary scattering problem. This solution (for the correct asymptotics at \( r \to \infty \)) is given by the Lippmann–Schwinger equation:

\[
\psi_k^{(i)}(\mathbf{r}) = e^{i \mathbf{k} \cdot \mathbf{r}} + \int G_0^{(i)}(E, \mathbf{r}, \mathbf{r}') V(\mathbf{r}') \psi_k^{(i)}(\mathbf{r}') \, d\mathbf{r}'
\]  

(1)

For \( r \to \infty \) the asymptotic is given:

\[
\left. \psi_k^{(i)}(\mathbf{r}) \right|_{r \to \infty} = e^{i \mathbf{k} \cdot \mathbf{r}} + \frac{e^{i \mathbf{k} \cdot \mathbf{r}}}{r} f(\mathbf{\omega}, \mathbf{\nu})
\]  

(2a)

where the scattering amplitude (which is directly related to the cross section) is

\[
f(\mathbf{\omega}, \mathbf{\nu}) = -\frac{\hbar}{2 \mu \hbar^2} \int e^{-i \mathbf{k} \cdot \mathbf{r}'} V(\mathbf{r}') \psi_k^{(i)}(\mathbf{r}') \, d\mathbf{r}'
\]  

(2b)

Let us solve Eqs. (2a) - (2b) by the iteration method:

\[
f(\mathbf{\omega}, \mathbf{\nu}) = -\frac{\hbar}{2 \mu \hbar^2} \int e^{-i \mathbf{k} \cdot \mathbf{r}'} V(\mathbf{r}') e^{i \mathbf{k} \cdot \mathbf{r}'} \, d\mathbf{r}'
\]

\[-\frac{\hbar}{2 \mu \hbar^2} \int \int e^{-i \mathbf{k} \cdot \mathbf{r}'} V(\mathbf{r}') G_0^{(i)}(E, \mathbf{r}, \mathbf{r}'' V(\mathbf{r}'') e^{i \mathbf{k} \cdot \mathbf{r}''} \, d\mathbf{r}' \, d\mathbf{r}'' +
\]

\[= \sum_n f_n^{(i)}(\mathbf{\omega}, \mathbf{\nu})
\]  

(3)

To get this equation we substitute (1) into Eq. (2b).

We will call \( f_n \) the scattering amplitude of the \( n \)-th order.

For the analysis of amplitudes it is useful to use Feynman diagram technique which was originally developed for the treatment of...
collision problems:

\[ f(\bar{u}', \bar{u}) = \frac{1}{2} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} G_0(\mathbf{p}, E, \bar{u}', \bar{u}) + \cdots \]

Let us agree on rules how to "read" the Feynman diagram:

- **(Solid line with free end)**: incoming or outgoing particle.
- **(Dashed line)**: interaction between particle and potential carrier.
- **(Solid line between two virtuals)**: propagator described by the free-particle Green's function.

Diagrams should be read in the direction opposite to the "direction of motion" of the particle.

2) If the interaction \( V(\mathbf{r}) \) is small (what this means mathematically we shall discuss a little bit later), we can restrict to the first-order amplitude:

\[
f(\bar{u}', \bar{u}) = \frac{\mu}{2\pi \hbar^2} \int e^{-i\mathbf{k} \cdot \mathbf{r}} \left[ e^{i\mathbf{h} \cdot \mathbf{r}} - 1 \right] \, d\mathbf{r} =
\]

\[
= \left\{ \text{just change variable of integration for simplifying} \right\} =
\]

\[
= -\frac{\mu}{2\pi \hbar^2} \int e^{-i\mathbf{h} \cdot \mathbf{r}} V(\mathbf{r}) e^{i\mathbf{k} \cdot \mathbf{r}} \, d\mathbf{r} \quad (4)
\]

This amplitude is said to be written in the **Born approximation**.

It can be seen that in the Born approximation, the amplitude \( f(\bar{u}', \bar{u}) \) depends on the momentum transfer (the difference of momenta...
\[ f(\tilde{u}, \tilde{u}') = -\frac{\mu}{2\hbar^2} \int e^{i\tilde{q}\cdot\tilde{r}} U(r) \, d \tilde{r}, \quad \tilde{q} = \tilde{u} - \tilde{u}' \] (5)

So, in the Born approximation the cross section of scattering process is given by:

\[ \frac{d\sigma}{d\Omega} = \left( \frac{\mu}{2\hbar^2} \right)^2 \left| \int e^{i\tilde{q}\cdot\tilde{r}} U(r) \, d \tilde{r} \right|^2 \] (6)

For the case of spherically symmetric potential, \( U(r) = U(r) \), we can further simplify the amplitude:

\[ f(\tilde{u}, \tilde{u}') = -\frac{\mu}{2\hbar^2} \int d\tilde{r} \int_{0}^{\infty} r^2 \, dr \int_{0}^{\infty} e^{iqr} r^2 \sin \theta \, d\theta U(r) \]

\[ = -\frac{2\mu}{\hbar^2} \int_{0}^{\infty} U(r) \cdot \sin qr \cdot r \, dr \] (7)

In this case the amplitude (and, hence, cross section) depends only on the module of transferred momentum!

3) Before we will use Born approximation to calculate physically relevant cross sections, let us discuss the range in which this approximation is valid.

For the evaluation of the Born amplitude we have substitute \( \psi_{\nu}'(r) \approx e^{i\tilde{q}r} \) into Eq. (26), which means that the second term in Eq. (4) is supposed to be much smaller than the first one:

\[ \left| \int G^{(4)} (E, \tilde{r}, \tilde{r}') U(\tilde{r}') \psi_{\nu}'(\tilde{r}') \, d \tilde{r}' \right| \ll \left| e^{i\tilde{q}\tilde{r}} \right| = 1 \] (8)

Let us simplify this expression:
By substituting in Eq. (8) exact solution by the plane wave and using explicit form of the Green's function we obtain:

\[ \frac{\mu^2}{2\pi^2} \left| \int \frac{e^{ik|r-r'|}}{r-r'} V(r') e^{ik|r'|} \, dr' \right| \ll 1 \quad (3) \]

Evaluation of this integral is very complicated even for simple potentials. Let us consider integral in Eq. (9) for \( r=0 \). Why we consider \( r=0 \)? It is clear that for \( r=0 \) the integrant is largest due to \( \frac{1}{r-r'} \) term and we restrict ourselves to the case of spherical potential:

\[ \frac{\mu}{2\pi^2} \left| \int e^{ikr+ikr'} \frac{V(r')}{r'} \, dr' \right| \ll 1 \quad (10a) \]

\[ \downarrow \]

\[ \frac{\mu}{2\pi^2} \left| \int e^{ik(r'+r)} (1+\cos \theta) \frac{V(r')}{r'} \, d\theta \, d\phi \, d\theta \, d\phi \right| \ll 1 \]

Let us remind that we assume for the moment potentials which act in the volume of final radius \( R \ll d \). In this sense, Coulomb potential is not short-range potential.

Let us now consider Eq. (10b) in two asymptotic cases:

If \( k\cdot d \gg 1 \) : (which means that the scattering length \( \lambda = \frac{k}{2} \) of the scattered particle is much smaller than the interaction region).

In this case:
we assume that in the volumetric case

\[ \frac{M}{\hbar^2 k^2} \left| \int_0^\infty \left( e^{2i\hbar r'} - 1 \right) V(r') \, dr' \right| = \sim \frac{M}{\hbar^2 k^2} \left( \int_0^d \left( e^{2i\hbar r'} - 1 \right) V(r') \, dr' \right) \]

\[ = \frac{M}{\hbar^2 k^2} \left| \int_0^d e^{2i\hbar r'} V(r') \, dr' - \int_0^d V(r') \, dr' \right| \approx \frac{M}{\hbar^2 k^2} \left( \int_0^d V(r') \, dr' \right) \cdot \omega \]

\[ \Rightarrow \frac{1}{d} \left| \int_0^d V(r') \, dr' \right| \ll \frac{\hbar^2 k^2}{\mu d} = \frac{\hbar^2 k^2}{2 \mu} \cdot \frac{2}{k} = \frac{E}{\omega d} \quad (11) \]

For \( kd < 1 \):

\[ e^{2i\hbar r'} - 1 \approx 4 + 2i\hbar r' + \ldots - 1 \approx 2i\hbar r' \]

Therefore:

\[ \frac{M}{\hbar^2 k^2} \left| \int_0^d 2i\hbar r' V(r') \, dr' \right| \ll 1 \]

\[ \Rightarrow \frac{1}{d^2} \left| \int_0^d V(r') \, dr' \right| \ll \frac{\hbar^2}{\mu d^2} \quad (12) \]

It is seen that condition (11) for \( kd \gg 1 \)

\[ \frac{1}{d} \left| \int_0^d V(r') \, dr' \right| \approx V_0 \ll E \quad \text{(where \( V_0 \approx V(r) \) in real)} \]

which is just assurance that incident electron energy should be larger than the interaction energy.

For the \( kd << 1 \), the requirement:

\[ \frac{1}{d^2} V_0 d^2 \ll \frac{\hbar^2}{\mu d^2} \Rightarrow V_0 \ll \frac{\hbar^2}{\mu d^2} \]

corresponds to the requirement that in the potential well of average potential \( V_0 \) and size \( d \)

has no bound solutions!

By finishing discussion at the applicability of Born approximation, we shall mention that Eqs. (11) - (12) don't allow us to answer...
another important question: how valid is the Born approximation, depends on the angle at which scattered particles is observed. We shall come back to this question later in the course.

Let us apply the Born approximation to study scattering of fast particles on the short-range potential. As the example of this potential we take Yukawa potential:

\[ V(r) = A \frac{e^{-r/a}}{r} \]  

(13)

By inserting (13) into Eq. (3):

\[ f(u, u') = -\frac{2\mu}{\hbar^2} \frac{1}{q} \int_0^\infty e^{-\frac{q}{2}} \sin q \cdot d\Gamma = \]

\[ = -\frac{2\mu}{\hbar^2} \frac{A^2 a^2}{1 + (qa)^2} \]  

(14)

Therefore, the differential cross section is given by:

\[ \frac{d\sigma}{d\Omega} = \frac{4\mu^2}{\hbar^4} A^2 a^4 \cdot \frac{1}{(1 + (qa)^2)^2} \]  

(15)

Where \( q = \vec{k} - \vec{k}' \) and therefore

\[ q = \sqrt{\vec{k}^2 + \vec{k}'^2 - 2\vec{k} \vec{k}' \cos \theta} = \langle k = k' \rangle = \]

\[ = k \sqrt{2 - 2 \cos \theta} = 2 k |\sin \frac{\theta}{2}| \]  

(16)

Therefore, the cross section is given finally:

\[ \frac{d\sigma}{d\Omega} = \frac{4\mu^2 A^2 a^4}{\hbar^4} \frac{1}{(1 + 4k^2 a^2 \sin^2 \frac{\theta}{2})^2} \]  

(17)
From Eq (17) it is seen that:

\[
\frac{d\sigma}{d\Omega} = \frac{4\mu^2a^4}{t^4} \cdot \frac{1}{(k^2 + (qa)^2)^2} \cdot \frac{d\gamma}{d\Omega} \quad (18)
\]

(ii) Maximum of angular distribution is at \( \theta = 0 \) (forward scattering).

For higher energies (higher \( k \))

the cross section decreases faster with angle.

So: for higher energies the emission is more forward directed.

Let us also calculate the total scattering cross section:

\[
\sigma = \int \frac{d\sigma}{d\Omega} \cdot d\Omega = \int \frac{4\mu^2a^4}{t^4} \cdot \frac{1}{(k^2 + (qa)^2)^2} \cdot \frac{d\gamma}{d\Omega} \quad (18)
\]

In place of integrating over momentum \( k' \), it is convenient to integrate Eq. (18) over the transferred momentum \( q \); from Eq. (16) we have:

\[
q^2 = 2k^2(1 - \cos \theta) = 2k^2(1 - \cos \theta)
\]

since \( k = k' = \text{const} \),

\[
2q dq = 2k^2 \sin \theta \, d\theta 
\]

\[
\Rightarrow 2q dq = 2k^2 \sin \theta \, d\theta 
\]

\[
\Rightarrow \sin \theta \, d\theta \cdot dq = 2\pi \frac{q \, dq}{k^2} 
\]

We used the axial symmetry of the system.

Therefore:

\[
\sigma = \int_{q_{\text{min}}}^{q_{\text{max}}} \frac{4\mu^2a^4}{t^4} \cdot \frac{1}{k^2(1 + (qa)^2)^2} \cdot \frac{q \, dq}{k^2} 
\]

It follows immediately from Eq. (16):

\[q_{\text{min}} = 0, \quad q_{\text{max}} = 2k\]

Therefore:

\[
\sigma = \frac{16\pi \mu^2a^4}{t^4} \cdot \frac{1}{1 + 4k^2a^2} 
\]
As seen from this expression, for the fast collisions, \( k \cdot a \gg 1 \), the cross section decreases as
\[
\sigma \sim \frac{1}{k^2} \sim \frac{1}{E}.
\]

**The features of the scattering on Yukawa potential (forward scattering, 1/E energy dependence) are the same for other potentials of finite radius.**

Based on the solutions for Yukawa potential, we can obtain Born approximation for scattering on the Coulomb potential. Really, \( V(r) = A \cdot \frac{e^{-\alpha r}}{r} \) as \( r \rightarrow \infty \) leads to \( A = \frac{1}{\alpha} \), \( A = 2.2 \cdot 2^2 \).

Therefore
\[
\begin{align*}
S_{c \rightarrow c} &= \frac{\langle \mu_c, h \rangle}{\langle L, h \rangle_{\alpha \rightarrow \infty}} = - \frac{2 \mu c_z c^2}{\hbar^2 q^2} \\
\end{align*}
\]

and the formula (15) for the differential cross section gives us the Rutherford formula:

\[
\frac{d\sigma}{d\Omega} = \frac{4 \mu_z^2 2^2 2^2 e^4}{16 \hbar^2 k^4 \sin^2 \frac{\theta}{2}} \tag{23}
\]

The total cross section \( \sigma \) can not be calculated from this expression, since the integrand diverges at small angles.

It is important to mention that Eq. (23) will be obtained also by solving the scattering problem for the Coulomb potential (in the following lecture).

The problem of scattering on the extended Coulomb potential is to be discussed later.
Problem 4

Scattered wave: $\psi_s = \frac{e^{ikr}}{r}$

The density current:

$$\overrightarrow{d} = \frac{i\hbar}{2\mu} \left( 4\overrightarrow{\nabla} \overrightarrow{\psi} - 4\overrightarrow{\psi} \overrightarrow{\nabla} \right) =$$

$$= \frac{i\hbar}{2\mu} \left( \frac{e^{ikr}}{r} \overrightarrow{\nabla} \frac{e^{ikr}}{r} - \frac{e^{-ikr}}{r} \overrightarrow{\nabla} \frac{e^{ikr}}{r} \right) =$$

$$= \frac{i\hbar}{2\mu} \left( \frac{e^{ikr}}{r} ( - \imath k \frac{e^{ikr}}{r} - \frac{1}{r^2} e^{-ikr} ) - \frac{e^{-ikr}}{r} ( \imath k \frac{e^{ikr}}{r} - \frac{1}{r^2} e^{-ikr} ) \right) =$$

$$= \frac{i\hbar}{2\mu} \left( -2\imath k \right) = - \frac{\hbar k}{r^2 \mu} \hat{r}$$

$\overrightarrow{D} f = \text{grad } f$ (points in the direction of greatest increase of $f$)

$\overrightarrow{D} \overrightarrow{a} = \text{div } \overrightarrow{a}$ (measure of a vector field increase)

For the case of incoming wave: $\psi_s = \frac{e^{-ikr}}{r}$

$$\overrightarrow{d} = \frac{i\hbar}{2\mu} \left( \frac{e^{-ikr}}{r} \overrightarrow{\nabla} \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r} \overrightarrow{\nabla} \frac{e^{-ikr}}{r} \right) =$$

$$= \frac{i\hbar}{2\mu} \left( \frac{e^{-ikr}}{r} \left( \imath k \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2} \right) - \frac{e^{ikr}}{r} \left( -\imath k \frac{e^{-ikr}}{r} - \frac{e^{-ikr}}{r^2} \right) \right) =$$

$$= \frac{i\hbar}{2\mu r^2} \left( 2\imath k \right) \hat{r} = - \frac{\hbar k}{r^2 \mu} \hat{r}$$