Quantum Mechanics II

Lecture 3

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Properties of angular momentum operators

In this chapter we will discuss two examples of angular momentum operators and their properties. First of all let us briefly recall the defining properties of such operators. Any operator $\hat{\boldsymbol{j}}$ that fulfills the commutation relation

$$[\hat{j}_i, \hat{j}_j] = \mathrm{i}\epsilon_{ijk}\hat{j}_k \tag{1}$$

we call angular momentum operator. For such an operator we can find a set of eigenfunctions $|jm\rangle$ with

$$\hat{\boldsymbol{j}}^2 |jm\rangle = j(j+1) |jm\rangle , \tag{2a}$$

$$\hat{j}_z |jm\rangle = m |jm\rangle , \tag{2b}$$

where $-j \leq m \leq j$ and we diagonalized $\hat{j}_z$ together with $\hat{\boldsymbol{j}}^2$. This is a choice, we could have chosen any other component of $\hat{\boldsymbol{j}}$.

Orbital angular momentum

Classically the orbital angular momentum vector $\boldsymbol{l}$ is given by the vector product

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}. \tag{3}$$

This is straightforwardly quantized by replacing $\boldsymbol{r}$ and $\boldsymbol{p}$ by their corresponding operators. This leads to the components of the orbital angular momentum operator

$$\hat{l}_x = -\mathrm{i}\left(y\partial_z - z\partial_y\right), \tag{4a}$$

$$\hat{l}_y = -\mathrm{i}\left(z\partial_x - x\partial_z\right), \tag{4b}$$

$$\hat{l}_z = -\mathrm{i}\left(x\partial_y - y\partial_z\right). \tag{4c}$$

In cartesian coordinates these formulas are not of great use for our purposes. By performing a standard coordinate transformation, we find the orbital angular momentum operator in spherical coordinates:

$$\hat{l}_x = \mathrm{i}\left(\sin\varphi\,\partial_\theta + \cot\theta\cos\varphi\,\partial_\varphi\right), \tag{5a}$$

$$\hat{l}_y = \mathrm{i}\left(-\cos\varphi\,\partial_\theta + \cot\theta\sin\varphi\,\partial_\varphi\right), \tag{5b}$$

$$\hat{l}_z = -\mathrm{i}\partial_\varphi. \tag{5c}$$

Correspondingly we find

$$\hat{\boldsymbol{l}}^2 = -\left[\frac{1}{\sin^2\theta}\partial_\varphi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta)\right]. \tag{6}$$

It is a bit tedious but straightforwardly checked that these operators fullfill the commutation relation of angular momentum operators.

As $\hat{\boldsymbol{l}}$ is an angular momentum operator, we want to find the set of $\hat{\boldsymbol{l}}^2$ and $\hat{l}_z$ eigenfunctions $|lm\rangle$ below. These functions $\langle\theta\varphi|lm\rangle = Y_{lm}(\theta,\varphi)$ can be found by solving the set of differential equations

$$\hat{\boldsymbol{l}}^2 Y_{lm}(\theta,\varphi) = l(l+1)Y_{lm}(\theta,\varphi), \tag{7a}$$

$$\hat{l}_z Y_{lm}(\theta,\varphi) = mY_{lm}(\theta,\varphi). \tag{7b}$$

For the solution we make the ansatz $Y_{lm}(\theta,\phi) = \Theta_{lm}(\theta)\Phi_m(\varphi)$. The φ-dependent function is immediately found from Eq. (7b), yielding

$$\hat{l}_z\Phi_m(\varphi) = m\Phi_m(\varphi), \tag{8}$$

and, therefore, $\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}m\varphi}$. The prefactor has been chosen such that the normalization condition

$$\int_0^{2\pi}\Phi^*_{m'}(\varphi)\Phi_m(\varphi)\mathrm{d}\varphi = \delta_{m'm} \tag{9}$$

is fullfilled. With $\Phi_m(\varphi)$ and our ansatz Eq. (7a) becomes

$$\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta_{lm}(\theta) = 0 \tag{10}$$

This equation can be solved by starting with $m = 0$ and then subsequently applying $\hat{l}_+$. Here, only the final result of this procedure shall be given:

$$\Theta_{lm}(\theta) = \frac{(-1)^m}{2^l l!}\left[\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}}\sin^m\theta\,\partial^{l+m}_{\cos\theta}(\cos\theta-1)^l. \tag{11}$$

Similarly to $\Phi_m(\varphi)$ this function fullfills the normalization condition

$$\int_0^{\pi}\Theta^*_{l'm'}(\theta)\Theta_{lm}(\theta)\sin\theta\mathrm{d}\theta = \delta_{l'l}\delta_{m'm}. \tag{12}$$

Note that $\Theta_{lm}(\theta)$ can only be found for integer values of m and, thus, only positive integers of l are allowed.

We call the functions $Y_{lm}(\theta,\phi)$ whose components have been derived above *spherical harmonics*, these are an irreducible representation of the $SU(2)$ or $SO(3)$ symmetry group.

Spin angular momentum

In this section we will discuss the special case of $j = \frac{1}{2}$. In this case we can find only two projections $m = \pm\frac{1}{2}$ and, thus, the eigenspace of $\hat{\boldsymbol{j}}^2$ and $\hat{j}_z$ is spanned by two vectors $|\frac{1}{2}\frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle$, only. A common choice to represent these as real two-vectors is

$$|{}^1/_2\, {}^1/_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |{}^1/_2, -{}^1/_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{13}$$

For this representation, we can find the spin-operator $\hat{\boldsymbol{s}}$ by requiring hermiticity and

$$\hat{s}_z\, |{}^1/_2\, \pm {}^1/_2\rangle = \frac{1}{2}\, |{}^1/_2\, \pm {}^1/_2\rangle \tag{14a}$$

$$[\hat{s}_i, \hat{s}_j] = \mathrm{i}\epsilon_{ijk}\hat{s}_k. \tag{14b}$$

We find that the spin-operator $\hat{\boldsymbol{s}}$ can be expressed in terms of Pauli matrices:

$$\hat{s}_x = \frac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{15a}$$

$$\hat{s}_y = \frac{1}{2}\sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{15b}$$

$$\hat{s}_z = \frac{1}{2}\sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15c}$$

Vector model

Classicaly angular momenta are described by 3-vectors in $\mathbb{R}^3$. We will use this section how the quantum-mechanical properties of angular momenta translate to this classical model. The length of a vector $\boldsymbol{r}$ is defined by $|\boldsymbol{r}| = \sqrt{\boldsymbol{r}\cdot\boldsymbol{r}} = \sqrt{\boldsymbol{r}^2}$. Correspondingly we find the length of the quantum mechanical angular momentum $\hat{\boldsymbol{j}}$ by

$$\langle jm|\hat{\boldsymbol{j}}^2|jm\rangle^{\frac{1}{2}} = \sqrt{j(j+1)}. \tag{16}$$

The direction of the vector $\hat{\boldsymbol{j}}$ is, correspondingly given by the expectation values of the components of $\hat{\boldsymbol{j}}$. For $\hat{j}_z$ it is $\langle\hat{j}_z\rangle = m$. But, since the components of $\hat{\boldsymbol{j}}$ do not commute, the x- and y-components of $\hat{\boldsymbol{j}}$ are not sharply defined. But we can calculate the variance of their values. The variance of an operator $\hat{A}$ is given by $\Delta_A = \langle A^2\rangle - \langle A\rangle^2$. For the case of $\hat{\boldsymbol{j}}$ we find:

$$\langle\hat{j}_{x/y}\rangle = 0 \tag{17a}$$

$$\langle\hat{j}_x^2 + \hat{j}_y^2\rangle = j(j+1) - m^2. \tag{17b}$$

Therefore the uncertainty of the direction of $\hat{\boldsymbol{j}}$ is minimal, if $m = \pm j$. But note that even if $j_z = m$, the angular momentum vector is not pointing along the z-axis. Instead it is precessing about this axis with a $\sqrt{j}$ deviation.