Exercise Sheet 2

This exercise was uploaded to the FPM homepage on Monday 27.04. Your solution must be sent as `02_name_surname.pdf` to `sebastian.ulbricht@ptb.de` until Sunday 10.05. The corrections and a sample solution will be sent to you on Tuesday 12.05. The exercise sheet will be discussed in the seminar on Thursday 14.05 at 9:45 a.m via dfn, following the link at Stud-IP.

4 Covariant Maxwell equations (10)

Assume the Lagrangian of the classical electromagnetic field

$$\mathcal{L}[A^\mu, \partial_\nu A^\mu] = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu$$

with the Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the four potential $A^\mu = (\Phi/c, \boldsymbol{A})$ and the four-current $j^\mu = (c\rho, \boldsymbol{j}\,)$.

a) Show that the Lagrangian above is invariant under gauge transformations $A_\mu \to A'_\mu = A_\mu + \partial_\mu \xi$, where $\xi(\boldsymbol{r}, t)$ is some scalar function, if the continuity equation $\dot{\rho} + \nabla \boldsymbol{j} = 0$ is satisfied.

b) Show that the Euler-Lagrange equations for A^μ have the form

$$\partial_\mu F^{\mu\nu} = -\mu_0\, j^\nu$$

c) and that they lead to the well known set of the four Maxwell equations for the electric and magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and $\boldsymbol{E} = -\nabla\Phi - \dot{\boldsymbol{A}}$.

5 The Lorentz group (5)

A Lorentz transformation $x^\mu \to x'^\nu = \Lambda^\mu{}_\nu x^\nu$ preserves the Minkowski line element. This implies that the metric tensor has to fulfill $\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$.

a) Use this result to show that an infinitesimal transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

is a Lorentz transformation if $\omega^{\mu\nu}$ is antisymmetric.

b) Write down a 4×4 matrix $\Omega_1 \sim (\omega^2{}_3)$, containing values of $1, 0$ and -1 that corresponds to an infinitesimal rotation about the z-axis. Similarly, construct the matrices Ω_2 and Ω_3 for boosts in the x- and y-direction, respectively.
Proof, that they form a Lie algebra, defined by the commutator relation

$$[\Omega_i, \Omega_j] = f^k_{ij}\,\Omega_k$$

and give the values of the structure constants f^k_{ij}.

c) Show that the exponential map of the Ω_i produces the matrix representations of the corresponding finite Lorentz transformations

(i) $\Lambda(\theta) \;= \exp(\theta\,\Omega_1)$ for a finite rotation with angle θ about the z-axis.

(ii) $\Lambda(v_x) = \exp(\sigma_x\,\Omega_2)$ for a x-boost with a finite velocity $v_x/c = \tanh(\sigma_x)$

(iii) $\Lambda(v_y) = \exp(\sigma_y\,\Omega_3)$ for a y-boost with a finite velocity $v_y/c = \tanh(\sigma_y)$

6 Properties of the functional derivative (5)

The functional derivative of a functional $F[\phi(\boldsymbol{r},t)]$ with respect to the field $\phi(\boldsymbol{r},t)$ is defined as

$$\frac{\delta F[\phi(\boldsymbol{r},t)]}{\delta\phi(\boldsymbol{r}',t)} = \lim_{\epsilon\to 0}\frac{F[\phi(\boldsymbol{r},t)+\epsilon\delta(\boldsymbol{r}-\boldsymbol{r}')]-F[\phi(\boldsymbol{r},t)]}{\epsilon} \quad .$$

Show that the functional derivative has the following properties:

a) $$\frac{\delta\phi(\boldsymbol{r},t)}{\delta\phi(\boldsymbol{r}',t)} = \delta(\boldsymbol{r}-\boldsymbol{r}')$$

b) $$\frac{\delta\phi(\boldsymbol{r},t)^n}{\delta\phi(\boldsymbol{r}',t)} = n\,\phi(\boldsymbol{r},t)^{n-1}\,\delta(\boldsymbol{r}-\boldsymbol{r}')$$

c) $$\frac{\delta}{\delta\phi(\boldsymbol{r}',t)}\Big(\nabla_{\boldsymbol{r}}\,\phi(\boldsymbol{r},t)\Big) = \nabla_{\boldsymbol{r}}\delta(\boldsymbol{r}-\boldsymbol{r}') = -\nabla_{\boldsymbol{r}'}\delta(\boldsymbol{r}-\boldsymbol{r}')$$

d) Now assume the functional $F[\phi(\boldsymbol{r},t)] = \exp\left(\int \phi(\boldsymbol{r},t) f(\boldsymbol{r},t)\mathrm{d}\boldsymbol{r}^3\right)$ and show that the functional derivative is given by

$$\frac{\delta F[\phi(\boldsymbol{r},t)]}{\delta\phi(\boldsymbol{r}',t)} = f(\boldsymbol{r}',t)F[\phi(\boldsymbol{r},t)] \quad .$$