Exercise Sheet 1

This exercise was uploaded to the FPM homepage on Monday 20.04. Your solution must be sent to `sebastian.ulbricht@ptb.de` until Sunday 26.04. The corrections and a sample solution will be sent to you on Tuesday 28.04. The exercise sheet will be discussed in the seminar on Thursday 30.04 at 9:45 a.m via dfn, following the link at Stud-IP.

1 Typical scales of QM and QFT (5)

Calculate the Compton wavelength and the rest mass energy of the electron and the up- and down-quark.

a) Compare the Compton wavelengths with typical length scales of the hydrogen atom and the proton. What is your conclusion concerning a quantum field theoretical treatment of the hydrogen atom or the proton?

b) Estimate the binding energy of the proton and the binding energy of the hydrogen atom (both in MeV). Compare them to the rest mass energy of the composite particle, respectively.

2 Causality (5)

Consider a relativistic particle at $\boldsymbol{r}_1$ at the time t_1. Show that in quantum mechanics the possibility to find the particle at a position $\boldsymbol{r}_2$, is non-zero also for arbitrary small travel times $ct := c(t_2 - t_1) \ll |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. Do that as follows:

a) Assume the Hamiltonian of a relativistic particle $\hat{H} = \sqrt{m^2c^4 + c^2\hat{\boldsymbol{p}}^2}$. Write down the amplitude $U_{12} = \langle \boldsymbol{r}_2, t_2 | \boldsymbol{r}_1, t_1 \rangle = \langle \boldsymbol{r}_2 | \mathrm{e}^{i\hat{H}t/\hbar} | \boldsymbol{r}_1 \rangle$ as an integral of momentum space and switch to spherical coordinates (p, θ_p, φ_p). Perform the angular integrals. Hint: Without loss of generality θ_p can be chosen to be the angle between $\boldsymbol{p}$ and $\boldsymbol{r}_2 - \boldsymbol{r}_1$.

b) Use the stationary phase method to show that the amplitude is evanescent but non-zero for $ct \ll |\boldsymbol{r}_1 - \boldsymbol{r}_2|$, i.e. outside the light cone and that it has the form

$$U_{12} \approx A\left(|\boldsymbol{r}_1 - \boldsymbol{r}_2|, t\right) \, \mathrm{e}^{-\frac{mc}{\hbar}\sqrt{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2 - c^2t^2}} \quad ,$$

where the $A\left(|\boldsymbol{r}_1 - \boldsymbol{r}_2|, t\right)$ has not to be given explicitly.

The stationary phase method is a tool to approximate integrals of the kind

$$\int_{-\infty}^{\infty} f(x)\mathrm{e}^{i\phi(x)}\mathrm{d}x \quad ,$$

where $f(x)$ is a slowly varying function. If the first derivative of $\phi(x)$ vanishes at $x = x_0$, a Taylor expansion of the phase $\phi(x) \approx \phi(x_0) + \phi''(x_0)(x - x_0)^2/2$ leads to

$$\int_{-\infty}^{\infty} f(x)\mathrm{e}^{i\phi(x)}\mathrm{d}x \approx \int_{-\infty}^{\infty} f(x_0)\mathrm{e}^{i\phi(x_0)}\mathrm{e}^{i\phi''(x_0)(x-x_0)^2/2}\mathrm{d}x = f(x_0)\mathrm{e}^{i\phi(x_0)}\sqrt{\frac{2\pi i}{\phi''(x_0)}} \quad ,$$

where the Gaussian integral was performed explicitly in the last step. For this exercise, there is no need to calculate $\phi''(x_0)$, since it only contributes to the amplitude $A\left(|\boldsymbol{r}_1 - \boldsymbol{r}_2|, t\right)$.

3 An effective field theory (10)

Assume a chain of N masses m coupled via springs of equal length a/N and spring constant k. Moreover, every mass is connected to its equilibrium position by a rubber band of linear elasticity σ.

a) Find the equations of motion by discussing the forces acting on every mass of the system. Express them in terms of the displacements q_i from equilibrium position of the i-th mass ($i = 1, \ldots, N$).

b) Show that the Lagrange function of the system is given by

$$L(t, \dot{q}_i, q_i) = \sum_{i=1} \frac{m}{2}\dot{q}_i^2 - \frac{k}{2}(q_{i+1} - q_i)^2 - \frac{\sigma}{2}q_i^2 \,.$$

c) Replace $q_i(t) = \phi(x_i, t)$, where $x_i = ia/N$ in the Lagrange function and perform the limit $N \to \infty$ (i.e $a/N \to \mathrm{d}x$), in order to obtain the *Lagrangian.* Keep the quantities $ka/N = k'$, $\sigma N/a = \sigma'$ and $mN/a = m'$ fixed.

d) Write down the action of the resulting field theory and obtain the equation of motion for the field $\phi(x, t)$ by applying the principle of stationary action, i.e variational principle.

e) Compare the result of (d) with the equation of motion for a relativistic heavy scalar field of mass M

$$(\hbar^2\partial_\mu\partial^\mu - M^2c^2)\phi(\boldsymbol{r}, t) = 0 \qquad\qquad (\textit{Klein-Gordon equation})$$

what are the quantities $(M/\hbar)_{\mathrm{eff}}$ and c_{eff} for the effective field theory that describes the chain of masses, assumed at the beginning?