Uncertainty of measurement based on calibration functions:

uninformative priors, Bayes' theorem and GUM Supplement 1

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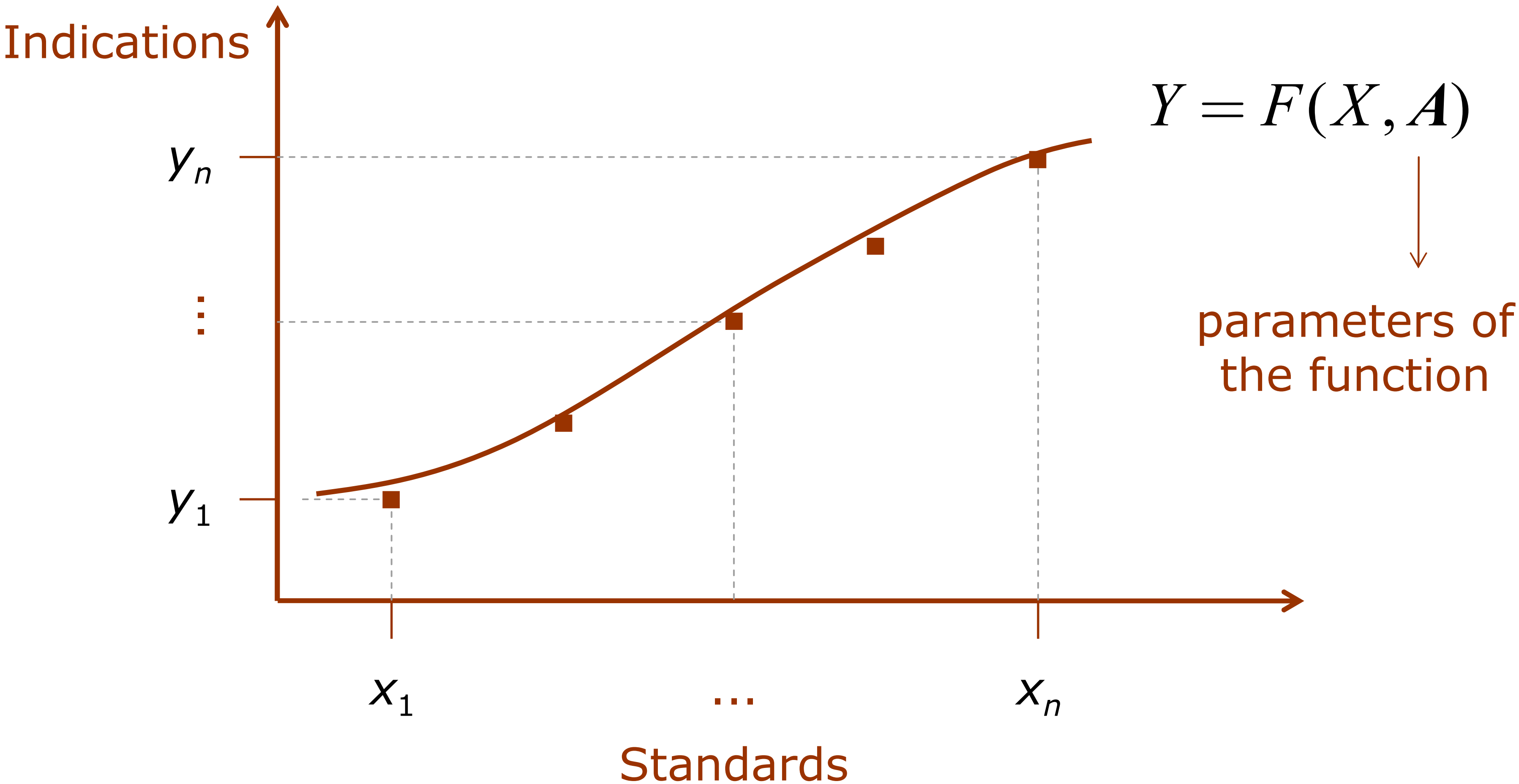
THE PROBLEM

An instrument is to be calibrated…

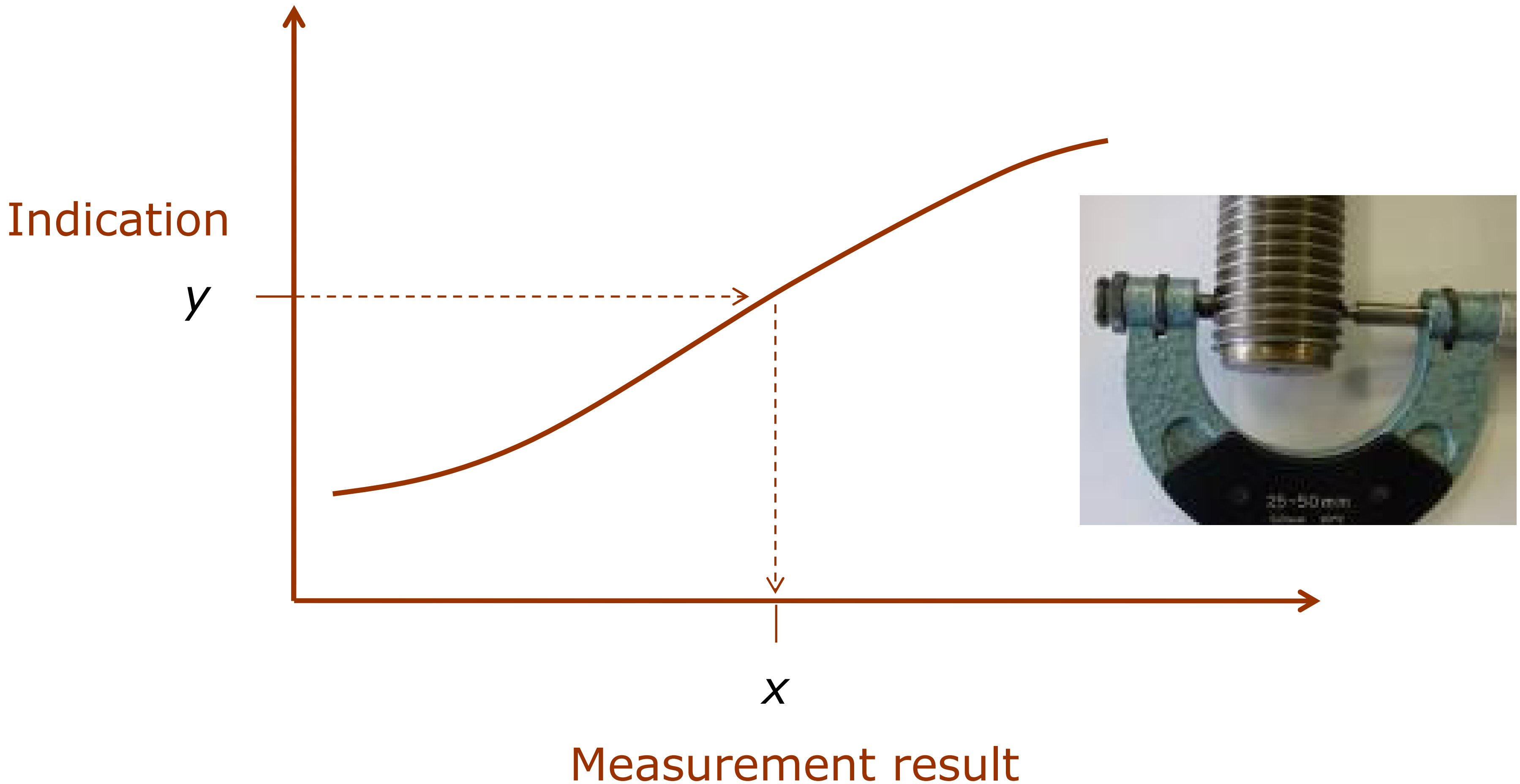
Standards covering the range of the instrument are chosen…

… and are "measured" with the instrument

A “calibration function” is constructed…



When the calibration function is put into practical use…



... the task is to obtain the posterior $f_X(\xi\,|y)$, from which the "best estimate" of measurement result x can be derived, together with the associated standard uncertainty $u(x)$.

Four steps can be distinguished:

1) Obtain a PDF $f_{\boldsymbol{A}}(\alpha)$ of the parameters $\boldsymbol{A}$

2) Establish the likelihood $l(\eta;y)$ of the indication y

3) Use Bayes' theorem to derive the posterior $f_Y(\eta\,|y)$

4) Use probability calculus to obtain the posterior $f_X(\xi\,|y)$

We shall assume that steps 1) and 2) have been completed

So only steps 3) and 4) will be adressed in the sequel

Step 3): Use Bayes' theorem to derive $f_Y(\eta \mid y)$

$$f_Y(\eta \mid y) \propto l(\eta; y) f_Y(\eta)$$

likelihood prior

Use non-informative prior if there is no information about Y

Selected by formal rules, e.g. those established by Bernardo and Berger

This type of prior is known as "reference" prior, and is considered by some as a "golden standard" in Bayesian analysis

Step 4): use probability calculus to derive $f_X(\xi \mid y)$

4.1 Independence between Y and $\boldsymbol{A}$

$$f_{Y,\boldsymbol{A}}(\eta, \boldsymbol{\alpha} \mid y) \propto f_Y(\eta \mid y) f_{\boldsymbol{A}}(\boldsymbol{\alpha})$$

4.2 Change of variables

$$f_{X,\boldsymbol{A}}(\xi, \boldsymbol{\alpha} \mid y) \propto f_{Y,\boldsymbol{A}}(F(\xi, \boldsymbol{\alpha}) \mid y) \left| \frac{\partial F(\xi, \boldsymbol{\alpha})}{\partial \xi} \right|$$

4.3 Marginalize

$$f_X(\xi \mid y) \propto \int f_{X,\boldsymbol{A}}(\xi, \boldsymbol{\alpha} \mid y) \, \mathrm{d}\boldsymbol{\alpha}$$

4.4 Normalize

4.5 Compute expectation and variance (if desired)

$$x = \mathrm{E}(X \mid y) \quad u^2(x) = \mathrm{V}(X \mid y)$$

GUM Supplement 1 is a numerical method to approximate the posterior $f_X(\xi\,|y)$

S.1 Draw a very large number of samples from $f_Y(\eta\,|y)$ and $f_{\boldsymbol{A}}(\alpha)$

S.2 Compute the corresponding values ξ of X by inverting the calibration function $Y=F(X,\boldsymbol{A})$

S.3 The frequency histogram of these values approximates $f_X(\xi\,|y)$

Within numerical accuracy, results should be exactly the same as the Bayesian approach described previously

➡ Bayesian analysis and GUM Supplement 1 are just two faces of the same coin

EXAMPLE

$$Y = A_1 + A_2 X + A_3 X^3$$

$$f_A(\boldsymbol{\alpha}) = \prod_{i=1}^{3} \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(\alpha_i - a_i)}{2\sigma_i^2} \right)$$

given estimates of A_1, A_2 and A_3

given variances

$$l(\eta; y) \propto \exp\left(-\frac{(\eta - y)^2}{2\sigma_y^2} \right)$$

$$f_Y(\eta) \propto 1$$

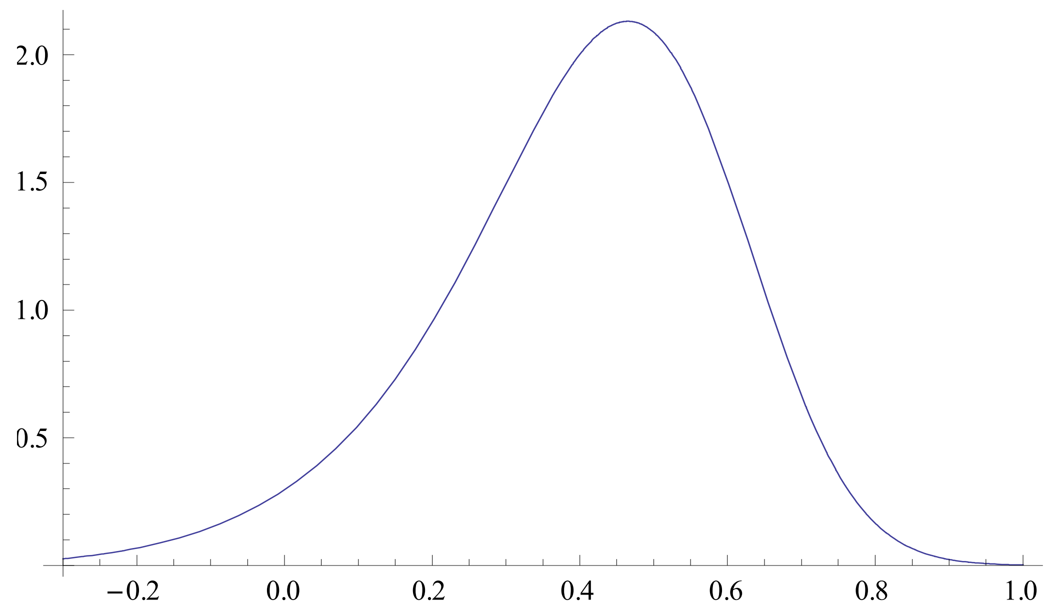
accepted reference prior for location parameter of Gaussian distribution

RESULTS

$y = 0.5$ $a_1 = 0$ $a_2 = a_3 = 1$ $\sigma_2 = \sigma_2 = \sigma_2 = \sigma_y = 1$

$f_X(\xi \mid y)$

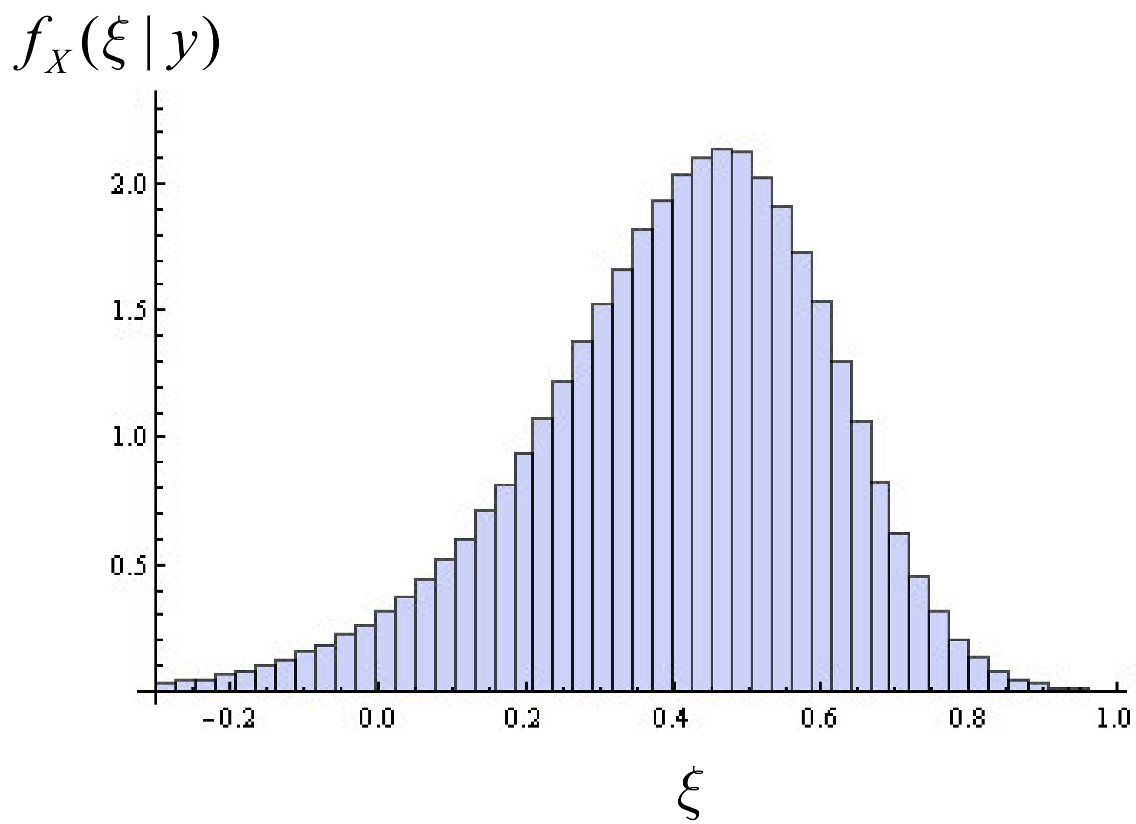
Bayesian analysis



ξ

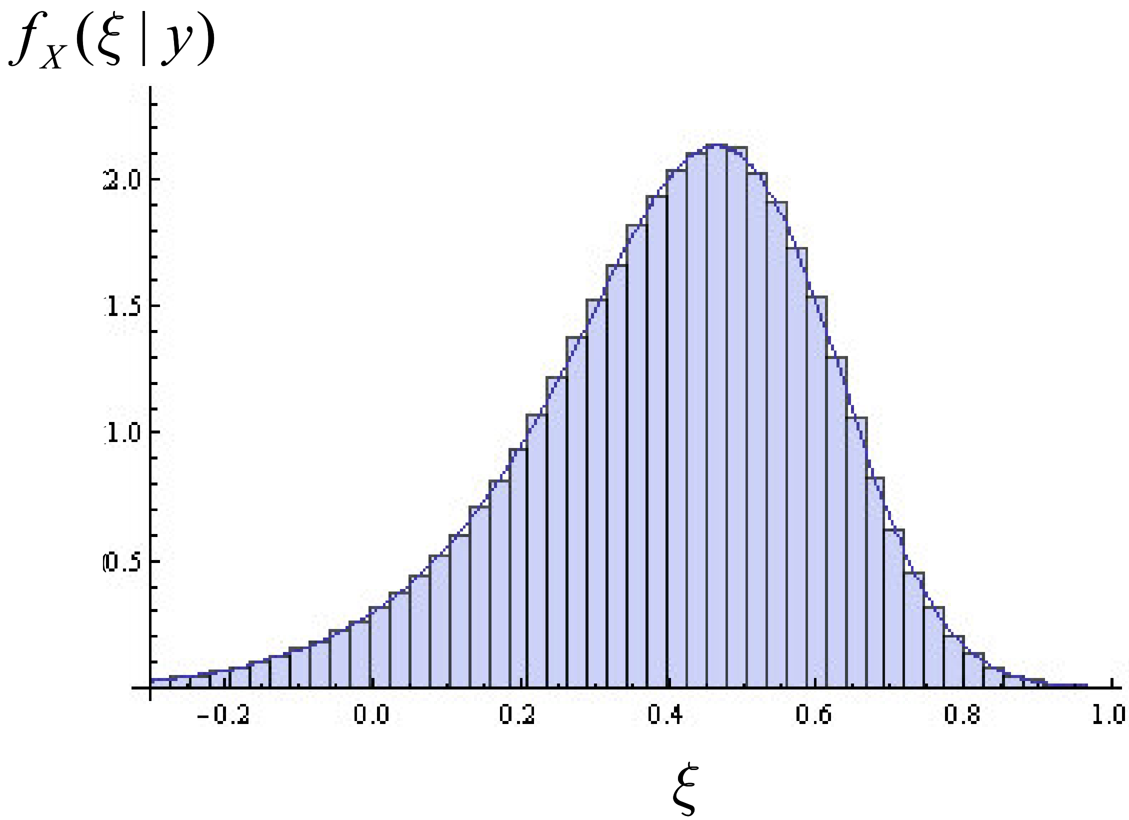
RESULTS

GUM S1
analysis



RESULTS

They both coincide



So this example corroborates that Bayesian and GUM S1 are equivalent procedures

Bayesian analysis can be carried out in an alternative way

Recall step 3

$$f_Y(\eta \mid y) \propto l(\eta; y) f_Y(\eta)$$

In this step, the likelihood can be written as

$$l(\eta; y) = l(F(\xi, \boldsymbol{\alpha}); y)$$

which prior to take?

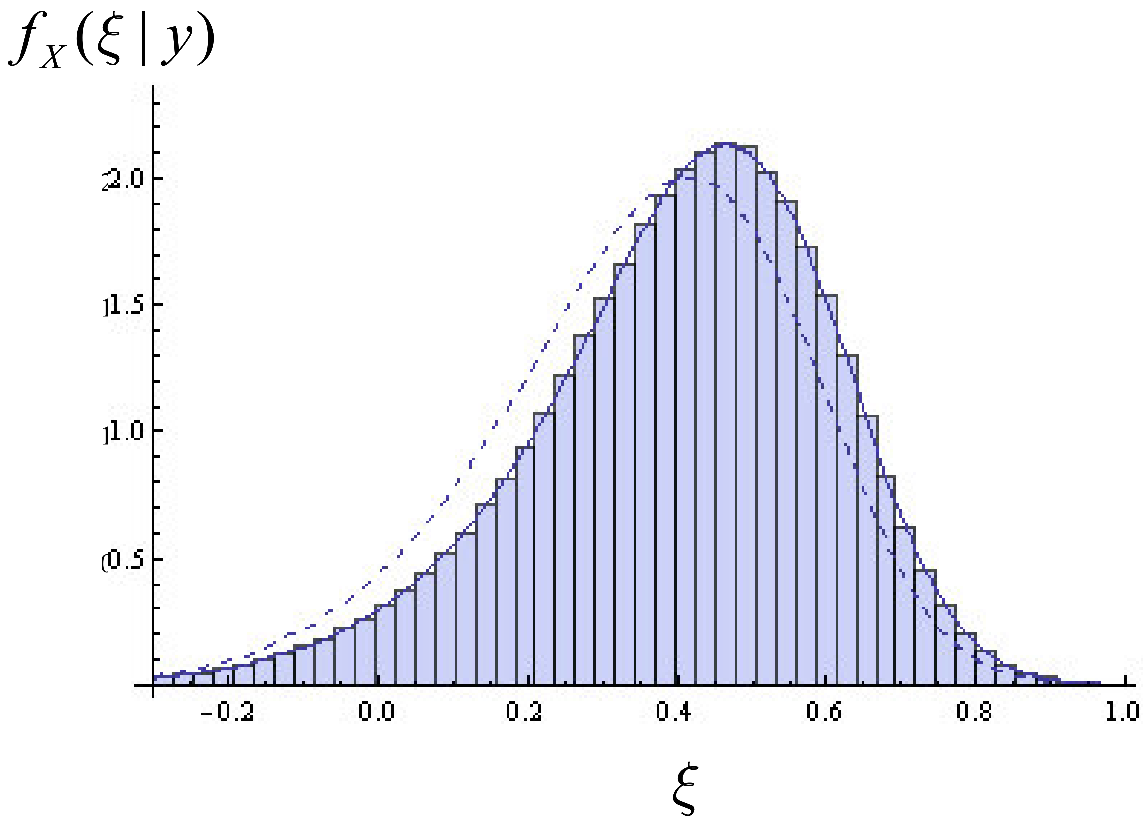
so $$f_{X,\boldsymbol{A}}(\xi, \boldsymbol{\alpha} \mid y) \propto l(F(\xi, \boldsymbol{\alpha}); y) f_X(\xi) f_{\boldsymbol{A}}(\boldsymbol{\alpha})$$

Since there is no information about *X*, take $f_X(\xi) \propto 1$

This choice results in the dashed curve

which differs from previous result

Both procedures are based on exactly the same (lack of) information, so they should coincide



The problem is that in the likelihood $l(F(\xi,\boldsymbol{\alpha});y)$ the quantity X is not a location parameter, so taking a uniform non-informative prior does not conform to the accepted rules for constructing a reference prior.

One should transform the uniform prior $f_Y(\eta)$

into a function of ξ and $\boldsymbol{\alpha}$:

$$f_Y(\eta) \to f_Y(F(\xi,\boldsymbol{\alpha}))\left|\frac{\partial F(\xi,\boldsymbol{\alpha})}{\partial \xi}\right| \propto \left|\frac{\partial F(\xi,\boldsymbol{\alpha})}{\partial \xi}\right|$$

and use this prior together with the likelihood expressed as a function of ξ and $\boldsymbol{\alpha}$:

$$f_{X,\boldsymbol{A}}(\xi,\boldsymbol{\alpha} \mid y) \propto l(F(\xi,\boldsymbol{\alpha});y)\left|\frac{\partial F(\xi,\boldsymbol{\alpha})}{\partial \xi}\right|$$

By doing so, the results of the alternative approach coincide with those of the original approach, and so, with those of the GUM S1

So, in conclusion, there should never be a difference between a Bayesian analysis and the Monte Carlo method in Supplement 1...

... provided consistent use is made of non-informative priors, if needed.

To avoid inconsistencies, these priors should be selected by formal rules based on the form of the likelihood.

END