Weighting observations from multi-station coordinate measuring systems

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Overview

Multi-station coordinate measuring systems

Posterior distribution for noise parameters

Maximising the posterior distribution

Application to laser tracker calibration

Large scale coordinate metrology

Target located at $\mathbf{x}$, station located at $\mathbf{p}$, orientation defined by rotation matrix $R(\boldsymbol{\alpha})$.

Distance to target given by $\|\mathbf{x}-\mathbf{p}\|$.

Azimuth and elevation angles: if $\mathbf{n} = \|\mathbf{x}-\mathbf{p}\|/d^*$ then there are $\theta^* = \theta^*(\mathbf{x},\mathbf{p},\boldsymbol{\alpha})$ and $\phi^* = \phi^*(\mathbf{x},\mathbf{p},\boldsymbol{\alpha})$ such that

$$\begin{bmatrix} \cos\theta^*\cos\phi^* \\ \sin\theta^*\cos\phi^* \\ \sin\phi^* \end{bmatrix} = R(\boldsymbol{\alpha})\mathbf{n}.$$

Observations

$$d = d^* + \epsilon_D, \quad , d^* = (1+\mu)\|\mathbf{x}-\mathbf{p}\| + \lambda, \quad \theta = \theta^* + \epsilon_A, \quad \phi = \phi^* + \epsilon_E,$$

involving random effects

$$\epsilon_D \in \mathrm{N}(0,\sigma_D^2), \quad \epsilon_A \in \mathrm{N}(0,\sigma_A^2), \quad \epsilon_E \in \mathrm{N}(0,\sigma_E^2),$$

due to refractive index changes, etc.

Multi-station systems

Target parameters $\mathbf{x}_j$, coordinates of the location of the jth target

Configuration parameters $\mathbf{b}$: locations $\mathbf{p}_k$, orientation angles $\boldsymbol{\alpha}_k$ associated with the measuring stations, other systematic effects λ_l, μ_r, etc.

Nonlinear least squares problem (MLE):

$$\min_{\{\mathbf{x}_j\},\mathbf{b}} \sum_i \left\{ w_{i,D}^2 (d_i - d_i^*(\mathbf{x},\mathbf{b}))^2 + w_{i,A}^2 (\theta_i - \theta_i^*(\mathbf{x},\mathbf{b}))^2 + w_{i,E}^2 (\phi_i - \phi_i^*(\mathbf{x},\mathbf{b}))^2 \right\}.$$

Weights $w_{i,D} = 1/\sigma_{i,D}$, etc.

Need to weight angle measurements relative to distance measurements.

Adjusting noise parameters: single sensor case

Suppose

$$\mathbf{y}|\boldsymbol{\alpha} \sim \mathsf{N}(C\boldsymbol{\alpha}, \sigma^2 I), \quad C, \quad m \times n, \quad y_i = \alpha_1 + \alpha_2 x_i + \epsilon_i, \quad \epsilon_i \in \mathsf{N}(0, \sigma^2).$$

Least squares solution:

$$\mathbf{a} = \left(C^\top C\right)^{-1} C^\top \mathbf{y} = R_1^{-1} Q_1^\top \mathbf{y}, \quad C = QR = [Q_1 \; Q_2] \left[\begin{array}{c} R_1 \\ \mathbf{0} \end{array} \right].$$

Also

$$\hat{\mathbf{y}} = C\mathbf{a} = Q_1 Q_1^\top \mathbf{y}, \quad \mathbf{r} = \mathbf{y} - \hat{\mathbf{y}} = (I - Q_1 Q_1^\top)\mathbf{y} = Q_2 Q_2^\top \mathbf{y},$$

so that

$$\mathbf{a}|\boldsymbol{\alpha}, \sigma \sim \mathsf{N}(\boldsymbol{\alpha}, \sigma^2 \left(C^\top C\right)^{-1}), \quad \mathbf{r}|\boldsymbol{\alpha}, \sigma \sim \mathsf{N}(\mathbf{0}, \sigma^2 Q_2 Q_2^\top).$$

If $\tilde{\mathbf{r}} = Q_2^\top \mathbf{r}$,

$$\tilde{\mathbf{r}}|\boldsymbol{\alpha}, \sigma \sim \mathsf{N}(\mathbf{0}, \sigma^2 I_{m-n}), \quad \frac{1}{\sigma^2}\mathbf{r}^\top \mathbf{r} = \frac{1}{\sigma^2}\tilde{\mathbf{r}}^\top \tilde{\mathbf{r}} \sim \chi^2_{m-n}.$$

Choose $\hat{\sigma}$ such the observed $\mathbf{r}^\top \mathbf{r}$ is $\mathsf{E}(\mathbf{r}^\top \mathbf{r}|\hat{\sigma})$:

$$\hat{\sigma}^2 = \frac{\mathbf{r}^\top \mathbf{r}}{m-n}, \quad \mathsf{E}(\hat{\sigma}|\sigma) = \sigma.$$

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Partial information about σ

Classical estimation caters for σ known or completely unknown. Suppose there is partial information about σ, e.g., from a previous experiment.

Suitable prior distribution for $\phi = 1/\sigma^2$:

$$m_0\sigma_0^2\phi \sim \chi^2_{m_0}, \quad p(\phi) \propto \phi^{m_0/2-1} \exp\left\{-\frac{\phi}{2}m_0\sigma_0^2\right\},$$

where σ_0 is the prior estimate and m_0 encodes degree of belief in σ_0. This prior corresponds to information gained about σ by observing a standard deviation of σ_0 of m_0 samples $z_i \in \mathrm{N}(0, \sigma^2)$.

Applying Bayes' theorem

$$p(\boldsymbol{\alpha}, \sigma|\mathbf{y}) \propto \phi^{(m_0+m)/2-1} \exp\left\{-\frac{\phi}{2}\left[m_0\sigma_0^2 + ms^2 + (\boldsymbol{\alpha} - \mathbf{a})^\top C^\top C(\boldsymbol{\alpha} - \mathbf{a})\right]\right\},$$

where

$$\mathbf{a} = (C^\top C)^{-1}C^\top\mathbf{y}, \quad \mathbf{r} = \mathbf{y} - C\mathbf{a}, \qquad ms^2 = \mathbf{r}^\top\mathbf{r}.$$

Marginal distributions

Model parameters:

$$p(\boldsymbol{\alpha}|\mathbf{y}) = \int_S p(\boldsymbol{\alpha}, \sigma|\mathbf{y}) d\sigma \Rightarrow \boldsymbol{\alpha} \sim t_\nu(\mathbf{a}, \bar{V}),$$

where

$$\nu = m_0 + m - n, \quad \bar{V} = \bar{\sigma}^2 (C^\top C)^{-1}, \quad \bar{\sigma}^2 = \frac{m_0 \sigma_0^2 + m s^2}{m_0 + m - n}.$$

If $m_0 \mapsto \infty$, $\bar{\sigma}^2 \mapsto \sigma_0^2$, if $m_0 \mapsto 0$, $\bar{\sigma}^2 \mapsto \widehat{\sigma}^2$.

Noise parameter:

$$p(\sigma|\mathbf{y}) = \int_A p(\boldsymbol{\alpha}, \sigma|\mathbf{y}) d\boldsymbol{\alpha} \Rightarrow \nu \bar{\sigma}^2 \phi \sim \chi_\nu^2.$$

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Multiple sensors

Suppose

$$\mathbf{y}_k|\boldsymbol{\alpha},\sigma_k \sim \mathsf{N}(C_k\boldsymbol{\alpha},\sigma_k^2 I), \quad k=1,\ldots,n_K,$$

where C_k is $m_k \times n$. Let

$$V_{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_1^2 I_{m_1} & & \\ & \ddots & \\ & & \sigma_{n_K}^2 I_{m_{n_K}} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ \vdots \\ C_{n_K} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_{n_K} \end{bmatrix}, \quad m = \sum_{k=1}^{n_K} m_k.$$

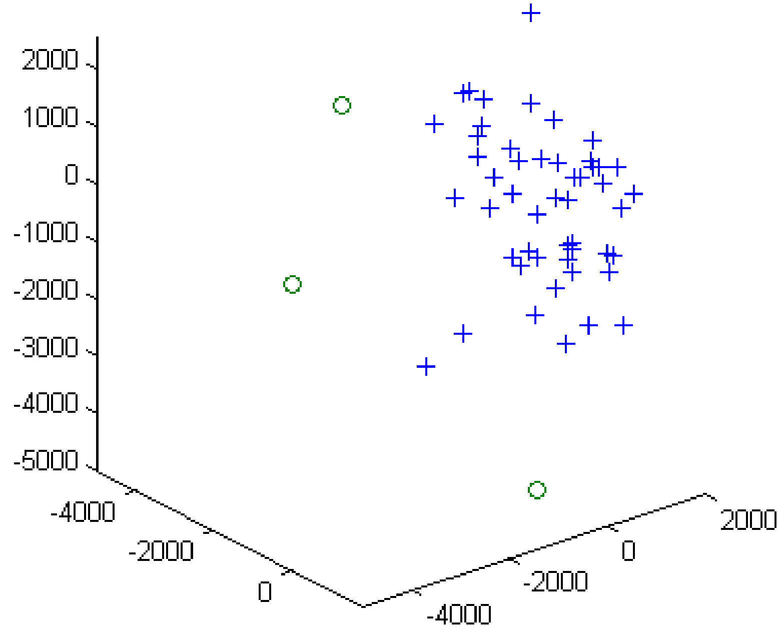
Naive posterior estimate of σ_k if $m_k > n$: given estimates $\sigma_{0,k}$, determine the least squares solution $\mathbf{a}$ of

$$\min_{\boldsymbol{\alpha}} (\mathbf{y} - C\boldsymbol{\alpha})^\top V_{\boldsymbol{\sigma}}^{-1}(\mathbf{y} - C\boldsymbol{\alpha}),$$

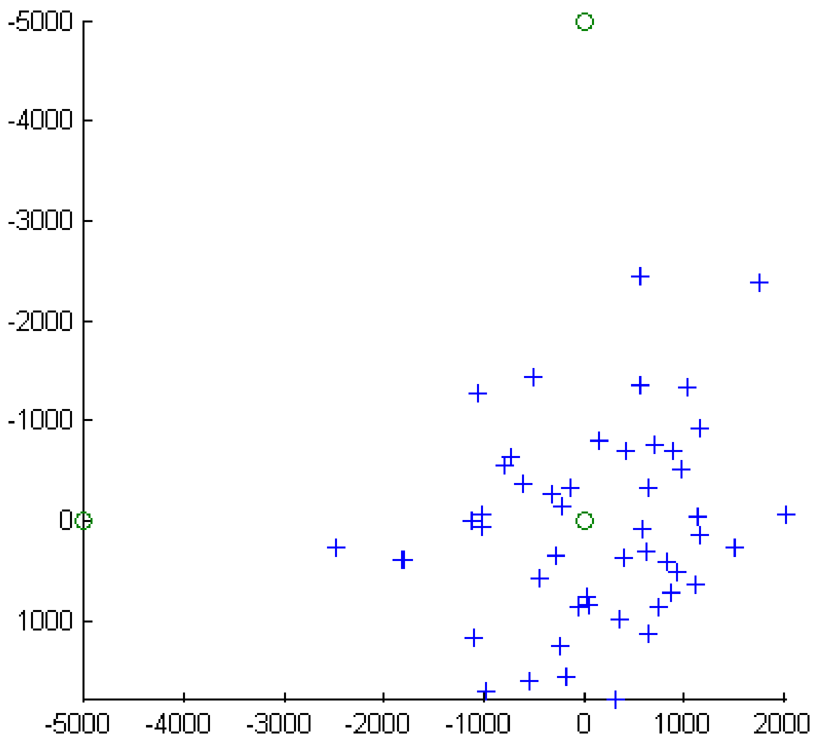
and set

$$\mathbf{r}_k = \mathbf{y}_k - C\mathbf{a}, \quad \hat{\sigma}_k^2 = \frac{m_k}{m}\frac{\mathbf{r}_k^\top \mathbf{r}_k}{m-n}.$$

Example: three laser trackers



Example: three laser trackers



Example: three laser trackers

$$\widehat{\sigma}_k^2 = \frac{m_k}{m}\frac{\mathbf{r}_k^\top \mathbf{r}_k}{m-n}$$

$$\widehat{\sigma}_D = 0.39, \quad \widehat{\sigma}_A = \widehat{\sigma}_E = 1.17.$$

Use these estimates to reweight, and recalculate $\widehat{\sigma}_k$:

$$\widehat{\sigma}_D = 0.13, \quad \widehat{\sigma}_A = \widehat{\sigma}_E = 1.05.$$

Better to compare $\mathbf{r}$ with $u(\mathbf{r})$ derived from $\mathbf{r}|\boldsymbol{\alpha}, \sigma \sim \mathrm{N}(\mathbf{0}, \sigma^2 Q_2 Q_2^\top)$, to take into account leverage (but ignoring correlation).

(ILCs: $C = (5,1,1,1,1,1,1,1)^\top$, $u(\mathbf{r}) = (0.26,1,1,1,1,1,1,1)^\top$. [Remember, $\mathbf{r}$ is the *weighted* residual.])

Multiple sensors, Bayesian approach

Prior information about $\phi_k = 1/\sigma_k^2$ given by

$$m_{0,k}\sigma_{0,k}^2\phi_k \sim \chi^2_{m_{0,k}}.$$

Assuming prior independence of $p(\phi_k)$,

$$p(\boldsymbol{\phi}) \propto \left(\prod_{k=1}^{n_K} \phi_k^{n_{0,k}/2-1}\right) \exp\left\{-\frac{1}{2}\sum_{k=1}^{n_K} n_{0,k}\sigma_{0,k}^2\phi_k\right\}.$$

Let

$$V_{\boldsymbol{\phi}} = \begin{bmatrix} \phi_1^{-1} I_{m_1} & & \\ & \ddots & \\ & & \phi_{n_K}^{-1} I_{m_{n_K}} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ \vdots \\ C_{n_K} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_{n_K} \end{bmatrix}, \quad m = \sum_{k=1}^{n_K} m_k.$$

If $p(\boldsymbol{\alpha}) \propto 1$, then the posterior distribution for $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ is such that

$$p(\boldsymbol{\alpha}, \boldsymbol{\phi}|\mathbf{y}) \propto p(\boldsymbol{\phi})|V_{\boldsymbol{\phi}}|^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{y} - C\boldsymbol{\alpha})^\top V_{\boldsymbol{\phi}}^{-1}(\mathbf{y} - C\boldsymbol{\alpha})\right\}.$$

Marginalised distribution

The marginalised distribution $p(\phi|\mathbf{y})$ is given by

$$p(\phi|\mathbf{y}) = \int p(\boldsymbol{\alpha}, \phi|\mathbf{y})\mathrm{d}\boldsymbol{\alpha}.$$

Let $\mathbf{a} = \mathbf{a}(\phi)$ solve

$$\min_{\boldsymbol{\alpha}} \quad (\mathbf{y} - C\boldsymbol{\alpha})^{\top} V_{\boldsymbol{\phi}}^{-1}(\mathbf{y} - C\boldsymbol{\alpha})$$

and set

$$(ms^2 = \mathbf{r}^{\top}\mathbf{r} =) \quad F(\phi) = (\mathbf{y} - C\mathbf{a})V_{\boldsymbol{\phi}}^{-1}(\mathbf{y} - C\mathbf{a}).$$

Then

$$p(\phi|\mathbf{y}) \propto p(\phi)|V_{\boldsymbol{\phi}}|^{-1/2}|C^{\top}V_{\boldsymbol{\phi}}^{-1}C|^{-1/2} \exp\left\{-\frac{1}{2}F(\phi)\right\}.$$

The ϕ that maximises $p(\phi|\mathbf{y})$ can be found by minimising $-\log p(\phi|\mathbf{y})$ or equivalently, minimising

$$E(\phi|\mathbf{y}) = \frac{1}{2}\left(F(\phi) + \log|V_{\boldsymbol{\phi}}| + \log|C^{\top}V_{\boldsymbol{\phi}}^{-1}C|\right) - \log p(\phi).$$

Evaluating $E(\boldsymbol{\phi}|\mathbf{y}) = \frac{1}{2}\left(F(\boldsymbol{\phi}) + \log|V_{\boldsymbol{\phi}}| + \log|C^\top V_{\boldsymbol{\phi}}^{-1}C|\right) - \log p(\boldsymbol{\phi})$

Firstly,

$$\log|V_{\boldsymbol{\phi}}| = -\sum_{k=1}^{n_k} m_k \log\phi_k.$$

Let

$$V_{\boldsymbol{\phi}}^{-1} = W_{\boldsymbol{\phi}}^2, \quad C_{\boldsymbol{\phi}} = W_{\boldsymbol{\phi}}C, \quad \mathbf{y}_{\boldsymbol{\phi}} = W_{\boldsymbol{\phi}}\mathbf{y}.$$

If $C_{\boldsymbol{\phi}}$ has QR factorisation

$$C_{\boldsymbol{\phi}} = [Q_1 \ Q_2]\begin{bmatrix} R_1 \\ \mathbf{0} \end{bmatrix} = Q_1R_1,$$

then

$$\log|C^\top V_{\boldsymbol{\phi}}^{-1}C| = \sum_{i=1}^{n} \log r_{ii}^2$$

where r_{ii} is the ith diagonal element of R_1, and

$$F(\boldsymbol{\phi}) = \mathbf{q}_2^\top\mathbf{q}_2, \quad \mathbf{q}_2 = Q_2^\top\mathbf{y}.$$

Evaluating $\partial E(\boldsymbol{\phi}|\mathbf{y})/\partial\phi_k$

Matrix determinant lemma: if A is square, full rank and a function of ϕ, then

$$\frac{d}{d\phi}\log|A| = \mathrm{Tr}\left[A^{-1}\frac{dA}{d\phi}\right].$$

Let

$$W_k = \frac{\partial W_{\boldsymbol{\phi}}}{\partial\phi_k} = \mathrm{diag}(\mathbf{0}, \mathbf{1}_k, \mathbf{0})^{\top}.$$

We have

$$\frac{\partial}{\partial\phi_k}\log|V_{\boldsymbol{\phi}}| = -\frac{m_k}{\phi_k},$$

$$\frac{\partial}{\partial\phi_k}\log|C^{\top}V_{\boldsymbol{\phi}}^{-1}C| = 2\mathrm{Tr}(S_k), \quad R_1 S_k = Q_1^{\top}W_k C$$

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Derivative calculations for least squares with matrices of functions

Let $m \times n$ matrix $C = C(\phi)$ be a function of a parameter ϕ and let $\dot{C} = dC/d\phi$ and

$$C = [Q_1 \; Q_2] \left[\begin{array}{c} R_1 \\ \mathbf{0} \end{array} \right], \quad \mathbf{a} = (C^\top C)^{-1} C^\top \mathbf{y} \quad \mathbf{r} = \mathbf{y} - C\mathbf{a}.$$

Then

$$\frac{d}{d\phi} \left[C(C^\top C)^{-1} C^\top \right] = F + F^\top, \quad F = Q_2 Q_2^\top \dot{C} R_1^{-1} Q_1^\top,$$

and

$$\frac{d}{d\phi}\mathbf{a} = (F + F^\top)\mathbf{y}, \quad \frac{d}{d\phi}\mathbf{r}^\top\mathbf{r} = -2\mathbf{r}^\top \dot{C}\mathbf{a}.$$

If $\mathbf{y}$ is also a function of ϕ, then

$$\frac{d}{d\phi}\mathbf{r}^\top\mathbf{r} = 2\mathbf{r}^\top \left(\dot{\mathbf{y}} - \dot{C}\mathbf{a} \right).$$

Application to $F(\phi)$

$$\frac{\partial F}{\partial \phi_k}(\phi) = 2(\mathbf{y}_{\boldsymbol{\phi}} - C_{\boldsymbol{\phi}}\mathbf{a})^{\top} W_k(\mathbf{y} - C\mathbf{a}) = 2\mathbf{q}_2^{\top} W_k W_{\boldsymbol{\phi}}^{-1}(Q_2\mathbf{q}_2).$$

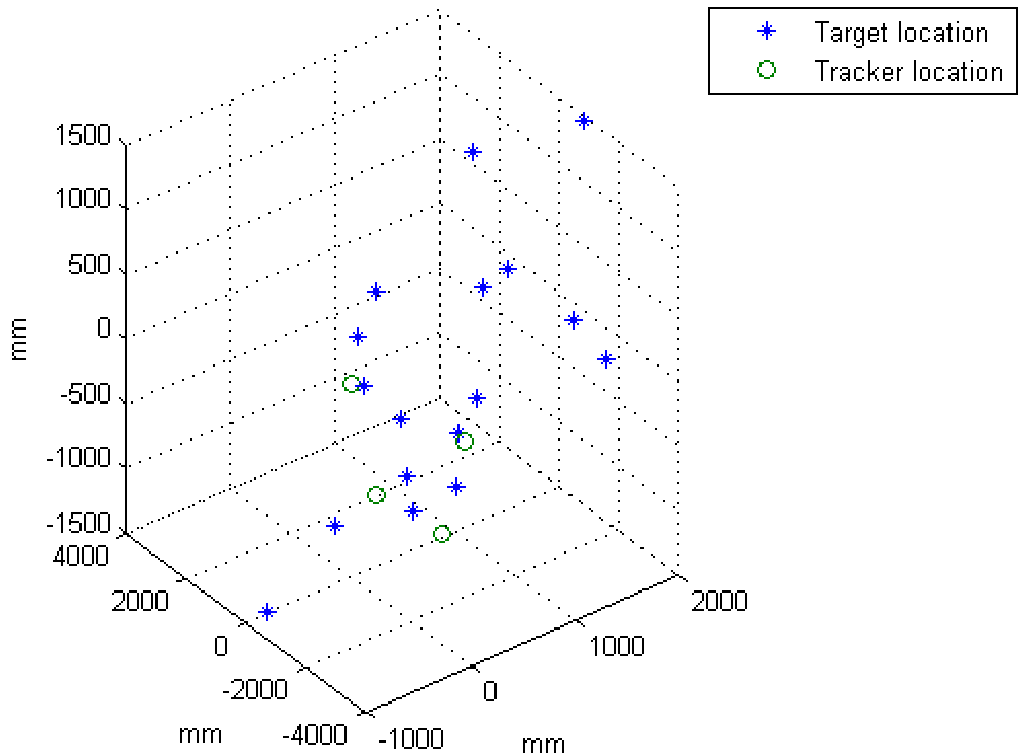
Partial upper-triangularisation

Let Q_k be such that

$$Q_k^{\top} \left[\begin{array}{cc} C_k & \mathbf{y}_k \end{array} \right] = \left[\begin{array}{cc} R_k & \mathbf{q}_k \\ & q_{0,k} \end{array} \right].$$

Then the least squares problem defined by $\{C_k, \mathbf{y}_k\}$ is equivalent to that defined by $\{R_k, \mathbf{q}_k, q_{0,k}\}$.

Tracker calibration experiments



Results

	sigma_0	sigma(Bayes)
distance	5.00e-3 mm	2.86e-3 mm
angle	5.00e-6 rad	4.31e-6 rad

Conclusions

Modal estimates of noise parameters obtained by maximising the marginalised posterior distribution $p(\boldsymbol{\phi}|\mathbf{y})$.

Applied to large scale coordinate metrology using laser trackers.

Calculations tractable using matrix factorisations, etc.

Posterior sampling algorithms: Markov chain Monte Carlo (Metropolis-Hastings with i) independent sampling, ii) random walks, iii) Hamiltonian (hybrid) Monte Carlo.

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