Supplement 1 to the GUM and Bayesian Inference

Clemens Elster

Physikalisch-Technische Bundesanstalt (PTB)

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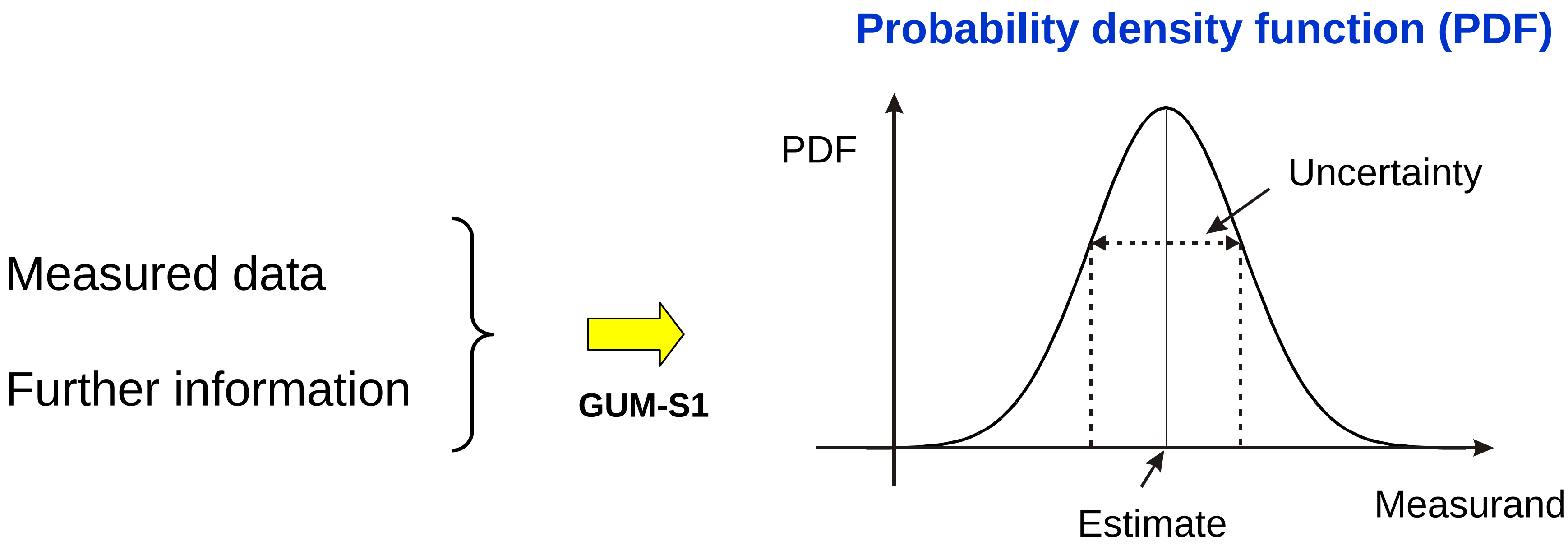
Content

- GUM-S1 and its relation to Bayesian inference

- Measurement model (GUM-S1) versus observation model

- GUM-S1 in the context of least-squares adjustment

GUM-S1 and Bayesian Inference



- GUM-S1 intends the Bayesian point of view
- But are GUM-S1 results equivalent to a standard Bayesian analysis?

GUM-S1

Input quantities

$X_1,\ldots,X_N$

Information

Type A / B

↓

$p(x_1,\ldots,x_N)$

Measurement model

$Y = f(X_1,\ldots,X_N)$

$Y = f(X_1,\ldots,X_N)$

→

Probability calculus

↑

Numerically: Monte Carlo method

Measurand

Y

$p(y)$

GUM-S1

Input quantities

$X_1, \ldots, X_N$

Measurement model

$Y = f(X_1, \ldots, X_N)$

Measurand

Y

Information

Type A / B

Inference

$p(x_1, \ldots, x_N)$

$Y = f(X_1, \ldots, X_N)$

Probability calculus

$p(y)$

Numerically: Monte Carlo method

Bayesian inference using observation model

Input quantities

$X_1, \ldots, X_N$

Measurement model

$Y = f(X_1, \ldots, X_N)$

Measurand

Y

Information

Type A / B

Inference

Bayes Theorem (type A info) via observation model[1]

$p(x_1, \ldots, x_N)$

$Y = f(X_1, \ldots, X_N)$

Probability calculus

$p(y)$

[1]Possolo and Toman 2007 **Metrologia** 44

GUM-S1 and Bayesian Inference

Within its specifications GUM-S1 is in the presence of type A info equivalent to standard (objective) Bayesian inference[1]

[1]Elster, Wöger and Cox 2007 **Metrologia** 44

GUM-S1 and Bayesian Inference

<u>Within its specifications</u> GUM-S1 is in the presence of type A info equivalent to standard (objective) Bayesian inference

<u>But:</u> Important features of Bayesian inference not addressed, e.g.

- Choice of prior, accounting for prior knowledge
- Sensitivity analysis w.r.t. (vague) prior
- Scenarios outside measurement model (e.g. least-squares)

Measurement vs. observation model

Given

$$Y = f(X, Z)$$ Measurement model

$$X = g(Y, Z)$$ Observation model

1) PDF $p(z)$

2) Data $x_1, \ldots, x_n \overset{\text{iid}}{\sim} \mathrm{N}(X, \Sigma^2)$

No prior knowledge

Goal Inference on *Y*

Measurement vs. observation model

Given

$$Y = f(X, Z)$$ Measurement model

$$X = g(Y, Z)$$ Observation model

1) PDF $p(z)$

2) Data $x_1, \ldots, x_n \overset{\text{iid}}{\sim} \mathrm{N}(X, \Sigma^2)$

Measurement model

$$\tilde{\boldsymbol{\theta}} = (X, Z, \Sigma)$$

$$p(\tilde{\boldsymbol{\theta}}) = p_{\tilde{\theta}}(x, z, \sigma)$$

$$\Longleftrightarrow$$

Observation model

$$\boldsymbol{\theta} = (Y, Z, \Sigma)$$

$$p(\boldsymbol{\theta}) = \left|\frac{\partial g}{\partial y}\right| p_{\tilde{\theta}}(g(y, z), z, \sigma)$$

Measurement vs. observation model

Given

$$Y = f(X, Z)$$ Measurement model

$$X = g(Y, Z)$$ Observation model

1) PDF $p(z)$

2) Data $x_1, \ldots, x_n \overset{\text{iid}}{\sim} \mathrm{N}(X, \Sigma^2)$

Prior $p(x, z, \sigma) \propto p(z)/\sigma$

$\tilde{\theta} = (X, Z, \Sigma)$ appropriate

$$p(x, \sigma) \propto 1/\sigma$$ Standard noninformative prior[1,2]

[1]Elster and Toman 2009 **Metrologia** 46, [2]Lira and Grientschnig 2010 **Metrologia** 47

Measurement vs. observation model

Given

$$Y = f(X, Z)$$ Measurement model

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1) PDF $p(z)$

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Prior $p(x, z, \sigma) \propto p(z)/\sigma$

Bayes theorem

Change-of-variables

$$p(y \mid x_1, \ldots, x_n) = \int |\partial g / \partial y| \, p(z) \frac{t_{n-1}[(g(y,z) - \bar{x})/(s/n^{1/2})]}{s/n^{1/2}} \mathrm{d}z$$

GUM-S1 PDF (Elster and Toman 2009 **Metrologia** 46)

Measurement vs. observation model

Given

$$Y = f(X, Z)$$ Measurement model

$$X = g(Y, Z)$$ Observation model

1) PDF $p(z)$ $p_0(y)$

2) Data $x_1, \ldots, x_n \overset{\text{iid}}{\sim} \mathrm{N}(X, \Sigma^2)$

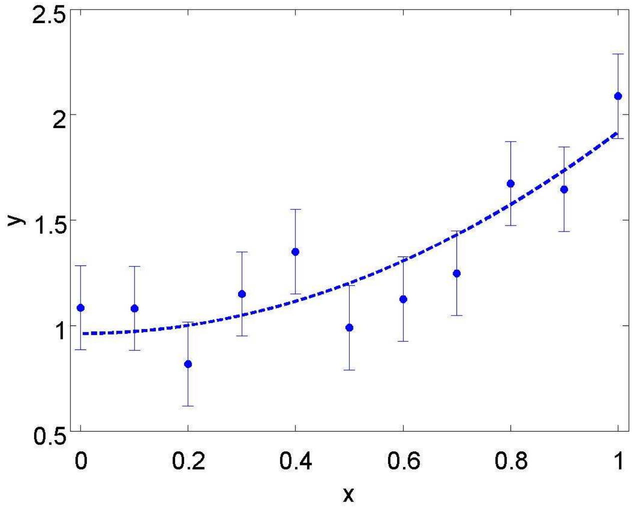
Prior $p(y, z, \sigma) \propto p_0(y)\, p(z) / \sigma$ $(\boldsymbol{\theta} = (Y, Z, \Sigma))$ appropriate

Bayes theorem → $p(y \mid x_1, \cdots, x_n) \neq p_{\text{GUM-S1}}(y)$

GUM-S1 choice of prior does not reflect prior knowledge

Least-squares adjustment

Data and least-squares estimate (dashed)



$$y_i = g_{\boldsymbol{\theta}}(x_i) + \varepsilon_i \qquad \varepsilon_i \overset{\text{iid}}{\sim} \mathrm{N}(0, \sigma^2) \quad (\sigma \text{ known})$$

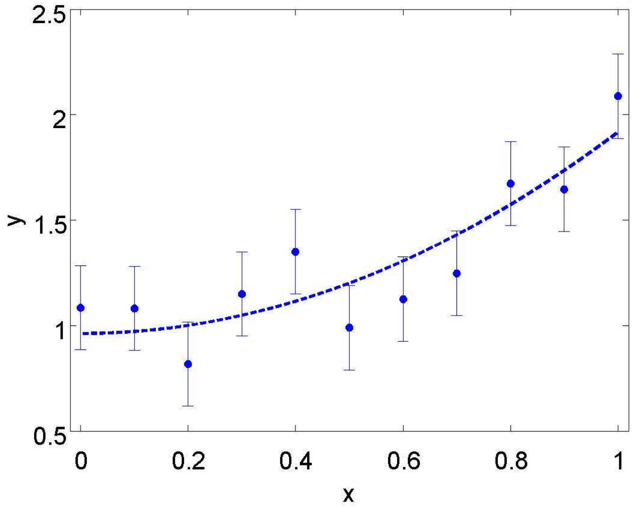
Goal: Estimation of $\boldsymbol{\theta}$

Least-squares estimate

$$\boldsymbol{\theta}_{\mathrm{LS}} = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} \frac{\left(y_i - g_{\boldsymbol{\theta}}(x_i)\right)^2}{\sigma^2}$$

Least-squares adjustment

Data and least-squares estimate (dashed)



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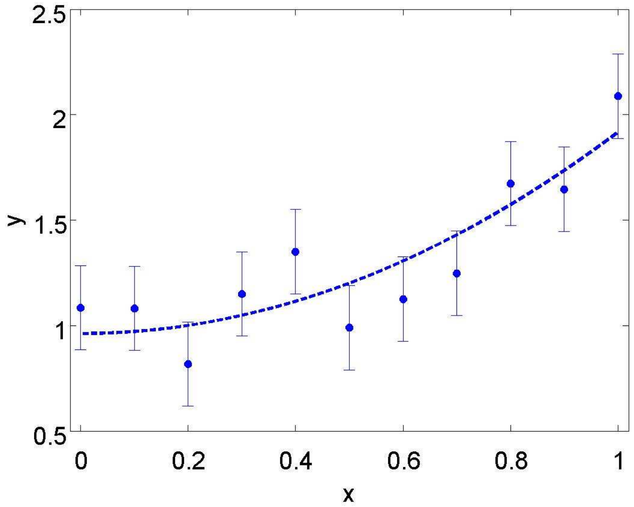
GUM-S1 ⟷ Bayesian inference ?

Least-squares adjustment

Data and least-squares estimate (dashed)



$$y_i = g_{\boldsymbol{\theta}}(x_i) + \varepsilon_i \qquad \varepsilon_i \overset{\text{iid}}{\sim} \mathrm{N}(0, \sigma^2) \quad (\sigma \text{ known})$$

Goal: Estimation of $\boldsymbol{\theta}$

Least-squares estimate

$$\boldsymbol{\theta}_{\mathrm{LS}} = \arg\min_{\theta} \sum_{i=1}^{n} \frac{\left(y_i - g_{\boldsymbol{\theta}}(x_i)\right)^2}{\sigma^2} \qquad (1)$$

Actually, (1) is not a measurement model between quantities, and is hence beyond scope of GUM S1. (It is the ML estimate). Nonetheless, GUM S1 can (and is) applied in this context.

Least-squares adjustment: GUM S1

Least-squares estimate $$\boldsymbol{\theta}_{\mathrm{LS}} = \arg\min_{\boldsymbol{\theta}} \chi^2(\boldsymbol{\theta}, \boldsymbol{y})$$

$$\chi^2(\boldsymbol{\theta}, \boldsymbol{y}) = \sum_{i=1}^{n} \frac{\left(y_i - g_{\boldsymbol{\theta}}(x_i)\right)^2}{\sigma^2}$$

GUM-S1: repeatedly draw $\tilde{\boldsymbol{y}} \sim \mathrm{N}(\boldsymbol{y}, \sigma^2 \boldsymbol{I})$

and determine $\tilde{\boldsymbol{\theta}}_{\mathrm{LS}} = \arg\min_{\boldsymbol{\theta}} \chi^2(\boldsymbol{\theta}, \tilde{\boldsymbol{y}})$

$\longrightarrow$ $p_{\mathrm{GUM-S1}}(\boldsymbol{\theta})$

Least-squares adjustment: Linear case

Bayesian posterior

$$p(\boldsymbol{\theta} \mid \text{data}) \propto p(\boldsymbol{\theta}) e^{-\chi^2(\boldsymbol{\theta},\boldsymbol{y})/2}$$

GUM-S1 PDF

$$p_{\text{GUM-S1}}(\boldsymbol{\theta}) \propto e^{-\chi^2(\boldsymbol{\theta},\boldsymbol{y})/2}$$

$$\boldsymbol{g}_{\boldsymbol{\theta}}(\boldsymbol{x}) = \boldsymbol{c}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\theta}$$

- GUM-S1 PDF equals Bayesian posterior for constant prior[1]

[1]Forbes (2009) **Advanced Mathematical & Computational Tools in Metrology VIII**, World Scientific

Least-squares adjustment: Nonlinear case

Bayesian posterior

$$p(\boldsymbol{\theta} \mid \text{data}) \propto p(\boldsymbol{\theta}) e^{-\chi^2(\boldsymbol{\theta}, \boldsymbol{y})/2}$$

GUM-S1 PDF

$$p_{\text{GUM-S1}}(\boldsymbol{\theta}) / p(\boldsymbol{\theta} \mid \text{data}) \neq g(\boldsymbol{\theta})$$

in general

GUM-S1 PDF and Bayesian posterior differ not just by a different choice of prior in general

Conclusions

- Within its specification GUM-S1 equivalent to (objective) Bayesian inference
- GUM-S1 prior corresponds to standard noninformative prior
- Observation model suitable for assignment of prior reflecting prior knowledge
- GUM-S1 for linear least-squares in the absence of prior knowledge is OK
- GUM-S1 for nonlinear least-squares does in general not yield a Bayesian posterior

Some references on this topic

- Lira I and Grientschnig D 2010 Equivalence of alternative Bayesian procedures for evaluating measurement uncertainty **Metrologia** 47 334-336
- Cox M G, Forbes A B, Harris P M and Smith I M 2009 Measurement uncertainty evaluations associated with calibration functions XIX **IMEKO** world congress 2009, Lisbon, Portugal
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- Forbes 2009 **Advanced Mathematical & Computational Tools in Metrology VIII**, World Scientific, 104-112
- Kyriazis G A 2008 Comparison of GUM Supplement 1 and Bayesian analysis using a simple linear calibration model **Metrologia** 45 L9-L11
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- Possolo A and Toman B 2007 Assessment of measurement uncertainty via observation equations **Metrologia** 44 464-475
- Kacker R, Toman B and Huang D 2006 Comparison of ISO-GUM, draft GUM supplement 1 and Bayesian statistics using simple linear calibration **Metrologia** 43 S167-S177
- Cox M G and Siebert B R L 2006 The use of a Monte Carlo method for evaluating uncertainty and expanded uncertainty **Metrologia** 43 S178-S188

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Thanks for your attention

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