Uncertainty as a Universal Aspect and an Advanced Approach for its Mathematical Handling

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What is Uncertainty?

Ambiguity of **uncertainty**; two main meanings:

- Uncertainty about the determinate past.
- Uncertainty about the indeterminate future.

Role of uncertainty:

- Major problem of mankind.
- Hardly considered in science.

Mathematical Theory of Uncertainty

Tasks to be solved:

- Unambiguous definition of uncertainty.
- Identification of the features relevant for uncertainty.
- Quantification of the relevant features by variables.
- Modelling the relations between the relevant variables.
- Developing means for applying the model of uncertainty.

Determinate Past

- Past characterized by facts.
- Uncertainty about facts = **ignorance**.
- Representation of ignorance = set of facts.
- Amount of ignorance = size of the set.
- Metrology:
  Attempts to determine facts
  =reducing the amount of ignorance
  = reducing the size of the corresponding set.

Indeterminate Future

- Future characterized by indeterminate events.
- Indeterminism of future due to **randomness**.
- Randomness = omnipresent physical phenomenon independent of man.
- Randomness follows exact laws.
- Metrology:
  Reducing ignorance by exploiting the laws of randomness.

Sources of Human Uncertainty about Future

- Randomness following certain laws.
- Laws of randomness determined by initial conditions.
- Ignorance about the initial conditions $\Rightarrow$ ignorance about laws of randomness $\Rightarrow$ increase in uncertainty about the future.
- Sources of uncertainty about future: **Randomness and ignorance**.

Randomness a Physical Quantity

- Randomness: property ascribed to future events $\Rightarrow$ randomness is a quantity.
- Physical quantity: quantity that can be used in the mathematical equations of science and technology[1] $\Rightarrow$ randomness is a physical quantity.
- Measurement of the value of a physical quantity assumes an adequate unit.
- No (base or derived) unit of randomness in the **International System of Units** (SI).

[1] According to: http://physics.nist.gov/cuu/Units/introduction.html.

Unit of Randomness

- Randomness: strength of attraction between initial condition and future events.
- Unit of randomness = **certainty** (by Jakob Bernoulli about 300 years ago).
- Lack of certainty about the occurrence of a future event = “degree of certainty ” of its occurrence = **probability**.
- Bernoulli: probability “differs from the certainty as part differs from the whole”.

Basics of a Measurement Procedures

- **Measurand**: Deterministic variable D with actual value d_0 representing an unknown fact.
- Measurement process (device):
  - Measurement range $\mathfrak{D}$ with $d_0 \in \mathfrak{D}$.
  - **Observational (random) variable** X.
- Relationship between D (past) and X (future): Set of **probability distributions** of the random variables $X|\{d\}$ (X under condition $\{d\}$) with $d \in \mathfrak{D}$:

$$\left\{ P_{X|\{d\}} \,\middle|\, d \in \mathfrak{D} \right\}$$

Probability Distributions

- Modelling randomness of $X|\{d\}$ = determining the probability distribution $P_{X|\{d\}}$.
- $P_{X|\{d\}}$ is a fact $\Rightarrow$ $P_{X|\{d\}}$ must be determined by measurements.
- Measurement processes $\Rightarrow$ $P_{X|\{d\}}$ (generally) uni-modal $\Rightarrow$ values of the quantities $E[X|\{d\}]$, $V[X|\{d\}]$, $\min X|\{d\}$ and $\max X|\{d\}$ needed .
- $E[X|\{d\}]$, $V[X|\{d\}]$, $Min\, X|\{d\}$ and $Max\, X|\{d\}$ determined based on copies of $X|\{d\}$ called sample $(X_1|\{d\}, X_2|\{d\}, \ldots, X_n|\{d\})$.

Calibration Function

- Calibration function: Relationship between d and the future (indeterminate) values of $X|\{d\}$, which will be observed.
- $X|\{d\}$ random variable $\Rightarrow$ no functional relationship between d and $X|\{d\}$.
- $E[X|\{d\}]$ deterministic variable $\Rightarrow$ functional relationship between d and $E[X|\{d\}]$. However: $E[X|\{d\}]$ is unobservable and unknown.
- Calibration function specifies the future, indeterminate outcome of $X|\{d\}$ $\Rightarrow$ calibration function = prediction procedure.

Stochastic Calibration and Measurement

- Stochastic calibration function = prediction procedure denoted $A_X^{(\beta)}$.
- $\{d\} \subset \mathfrak{D} \Rightarrow A_X^{(\beta)}(\{d\})$ with

$$P_{X|\{d\}}\left(A_X^{(\beta)}(\{d\})\right) \geq \beta$$

  where β is called **reliability level**.
- Stochastic measurement procedure denoted $C_D^{(\beta)}$ by stochastic inversion of $A_X^{(\beta)}$:

$$C_D^{(\beta)}(\{x\}) = \left\{d \,\middle|\, x \in A_X^{(\beta)}(\{d\})\right\} \subset \mathfrak{D}$$

A Simple Illustrative Example

Consider a spring scale to measure the weight of a given object. Any measurement based on two variables and one process:

- Deterministic variable (measurand) D = weight of an object
- Measurement process: Extension of a spring by a force.
- The random variable (observational variable) X = spring extension.

Determination of the Probability Distributions $P_{X|\{d\}}$

1. Range of measurement $\mathfrak{D}$ and standard weights d_k, $k = 1, \ldots, m$.
2. Samples of size n (n copies of $X|\{d_k\}$) $(X_1|\{d_k\}, \ldots, X_n|\{d_k\})$, $k = 1, \ldots, m$.
3. Measurement procedures with specified reliability levels for $E[X|\{d_k\}]$, $V[X|\{d_k\}]$, $\min X|\{d_k\}$ and $\max X|\{d_k\}$ yielding measurement intervals for each of the four parameters.
4. Extension of results for d_k, $k = 1, \ldots, m$, to cover $\mathfrak{D}$.
5. Determination of $\{P_{X|\{d\}}\}$ for $d \in \mathfrak{D}$ by means of an algorithm.

Calibration Function $A_X^{(\beta)}$

- Fixing the reliability level β.
- Calibration function (prediction procedure) $A_X^{(\beta)}$ meeting the reliability requirement β and minimizing the amount of ignorance each for $d \in \mathfrak{D}$:

$\{d\}$ [g]	$\{5.00\}$	$\{5.05\}$
$A_X^{(\beta)}(\{d\})$ [cm]	$\{x \vert 0.400 \leq x \leq 0.500\}$	$\{x \vert 0.433 \leq x \leq 0.533\}$

$\{d\}$ [g]	$\{5.10\}$	$\{5.15\}$
$A_X^{(\beta)}(\{d\})$ [cm]	$\{x \vert 0.467 \leq x \leq 0.567\}$	$\{x \vert 0.500 \leq x \leq 0.600\}$

Measurement Procedure $C_D^{(\beta)}$

- Measurement results by stochastic inversion of $A_X^{(\beta)}$:

$$C_D^{(\beta)}(\{x\}) = \left\{ d \in \mathfrak{D} \mid x \in A_X^{(\beta)}(\{d\}) \right\}$$

$\{x\}$ [cm]	$\{0.500\}$	$\{0.505\}$
$C_D^{(\beta)}(\{x\})$ [g]	$\{d \mid 5.00 \leq d \leq 5.15\}$	$\{d \mid 5.02 \leq d \leq 5.17\}$
$\{x\}$ [cm]	$\{0.510\}$	...
$C_D^{(\beta)}(\{x\})$ [g]	$\{d \mid 5.04 \leq d \leq 5.19\}$	...

Properties of the stochastic measurement procedure:
Reliability requirement met, amount of ignorance minimum.

Stochastic and GUM Measurements Procedures: A Comparison

- Concept of probability:
  - Stochastics: Base unit of randomness.
  - GUM: Ambiguous and unclear definition.
- Probability distributions:
  - Stochastics: Result of stochastic measurement procedures.
  - GUM: Standard probability distribution with estimated parameter values.
- Uncertainty:
  - Stochastics: Probability distributions of $X|\{d\}$, $d \in \mathfrak{D}$.
  - GUM: Standard deviation of estimator of D.

Further Differences

- Errors:
  - Stochastics: No errors considered.
  - GUM: Systematic error, random error.
- Classification of results:
  - Stochastics: Correct results, wrong results.
  - GUM: No classification.
- Reliability of measurement procedure:
  - Stochastics: Specified by the reliability level β.
  - GUM: Unspecified.

Further Differences

- Representation of measurement procedure:
  - Stochastics: Measurement procedure given by all measurement results $C_D^{(\beta)}(\{x\})$.
  - GUM: No complete representation possible, since amount of ignorance (= measurement uncertainty) calculated only afterwards.
- Measurement precision and accuracy:
  - Stochastics: Not applicable.
  - GUM: Precision estimated, accuracy not assignable.

Conclusions

- Metrology is based on the International System of Units (SI), which does not contain a unit of randomness.
- The unavailability of a unit of randomness rules out a meaningful consideration of uncertainty.
- Science, technology and metrology would benefit greatly, if the International System of Units would be completed by a base unit of randomness.
- More information about stochastics and how to handle uncertainty may be obtained on:

http://www.stochastikon.com