Computational Tools for PDEs with Random Coefficients

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ABSTRACT

The simulation and forecast of complex physical processes in science, engineering and industry requires input data which are often subject to considerable uncertainties. This is due to incomplete models, measurement errors or lack of knowledge. Partial differential equations (PDEs) with random coefficients offer the opportunity to incorporate data uncertainties in mathematical models and subsequent computer simulations. However, these PDEs are formulated in a physical domain coupled with a sample space generated by random parameters and can be very computing-intensive. Depending on the statistical properties of the input data the sample space can also be very high-dimensional. In the last decade there has been a lot of research activity in the applied mathematics and computational science communities to develop efficient computational tools for PDEs with random coefficients.

We outline the key computational challenges by discussing a model elliptic PDE of single phase subsurface flow in random media. In this application the coefficients are often rough, highly variable and require a large number of random parameters. To date, only Monte Carlo based methods are computationally feasible for this application since Monte Carlo methods decouple the physical and stochastic degrees of freedom. We employ multilevel Monte Carlo (MLMC), a novel variance reduction technique, see (Heinrich, 2001) and (Giles, 2008), which has recently been applied successfully to PDEs with random coefficients (Barth et al., 2011; Cliffe et al., 2011; Teckentrup et al., 2013). We explain the basic MLMC idea and combine this technique with mixed finite element discretisations to calculate travel times of particles in groundwater flows. As a second example, we will outline a novel multilevel estimator for rare events based on subset simulation (Au & Beck, 2001).

For coefficients which can be parameterised by a small number of random variables we employ spectral stochastic Galerkin (SG) methods (Ghanem & Spanos, 1991) which give rise to a coupled system of deterministic PDEs. Since the standard SG formulation of the model elliptic PDE requires expensive matrix-vector products we reformulate it as a convection-diffusion problem with random convective velocity (Ullmann et al., 2012). We construct fast iterative solvers for the nonsymmetric Galerkin matrix and present some performance tests.

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