Analytical study of the magnetic field from extended sources in subcortical structures

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1 Introduction

Current sources within the brain generate weak magnetic fields outside the head [1]. These magnetic fields depend on the location and the features of the sources. The purpose of the present study is to investigate the effects of extended sources located in deep brain structures, such as the midbrain or brainstem, by analytical model calculations.

For the sources two line source configurations of different properties are considered. The line sources are assumed to be located deep in a conducting sphere of constant conductivity used to model the influence of the head.

The magnetic field outside the spherical head model is studied by means of multipole expansion. This expansion is represented in terms of magnetic multipole moments at the center of the conducting sphere. The information content of the magnetic multipole fields is analyzed up to the octupole term for locations of the line sources near the center of the head model. As a special case a location of the source models at the center of the conducting sphere is considered. For this situation the magnetic field of a current dipole source is known to vanish [1, 2]. The results allow general conclusions concerning the magnetic field of sources at the center of a conducting sphere.

In addition to the forward problem, multipole expansion is also used for inverse calculations. For both source models the single current dipole solution of the inverse problem is derived analytically.

2 Methods

The magnetic field outside a conducting sphere is represented as the sum of the source term and the contribution of the volume conductor [1, 2, 3]. For the field of the source described by a primary current density a spherical harmonic multipole expansion is performed. The origin of the coordinate system is taken as the center of the conducting sphere. The expansion of the source term is characterized by two types of multipole coefficients which can be classified as the electric moments A_{nm}, B_{nm} and the magnetic moments α_{nm}, β_{nm} [4, 5, 6, 7]. The magnetic field contributions of the electric multipole moments and the conducting sphere cancel each other. Therefore, the magnetic field is given by the expansion

$$\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}\operatorname{Re}\sum_{n=1}^{\infty}\sum_{m=0}^{n}(\alpha_{nm} + i\beta_{nm})\nabla\left[\frac{\overline{Y}_n^m(\vartheta,\varphi)}{r^{n+1}}\right] \quad (1)$$

with the magnetic multipole moments

$$\alpha_{nm} + i\beta_{nm} = \frac{g_{nm}}{n+1}\int_V [\vec{r}\times\vec{J}^P]\nabla[r^n Y_n^m(\vartheta,\varphi)]dV. \quad (2)$$

In these expressions, $\vec{B}$ is the magnetic induction, and μ_0 is the permeability of free space. Re denotes the real part, i is the imaginary unit, ∇ is the del operator, and r, ϑ, φ are the spherical coordinates. Further, $Y_n^m(\vartheta,\varphi) = P_n^m(\cos\vartheta)e^{im\varphi}$ are the spherical harmonics, where $P_n^m(\cos\vartheta)$ are the associated Legendre functions of the first kind. The bar over $Y_n^m(\vartheta,\varphi)$ denotes the complex conjugate. $\vec{J}^P$ represents the primary current density. In addition, $g_{nm} = (2-\delta_{m0})(n-m)!/(n+m)!$, where δ_{m0} is the Kronecker delta which is unity for $m = 0$ and zero otherwise. The index n gives the order of the multipole. Note that $n = 1$ is the magnetic dipole, $n = 2$ the magnetic quadrupole, $n = 3$ the magnetic octupole, etc.

The effects of subcortical sources on the magnetic field outside the conducting sphere are studied for two line source types. Model LS1 is a line carrying a constant current I pointing in x direction of the Cartesian coordinates. The spatial extent of this line current is chosen to be $2a$. Line source type LS2 of extent $2b$ has a uniform distribution of perpendicular current dipoles pointing in x direction and is parallel to the y axis. The geometrical center of both sources is assumed to be located at z_0 on the z axis. These source types represent the continuous versions of the in-line source and the side-by-side line source, respectively, of the numerical study [8] on EEG and MEG dipole inverse solutions for various cortical sources in a spherical head model.

In addition to the magnetic field calculations, multipole expansion is also used to determine the single current dipole solution of the biomagnetic inverse problem. To obtain this equivalent current dipole (ECD) a method developed for the electric potential [9, 10] is applied to the magnetic field. The ECD is considered to be a source which generates the same magnetic dipole and quadrupole fields of the multipole expansion like the actual line sources located near the center of the conducting sphere. The higher-order terms beyond the magnetic quadrupole are neglected. The ECD parameters are obtained from a system of equations resulting from the magnetic dipole and quadrupole moments of the multipole expansion. In the present case this system reduces to the equations for an ECD of the current dipole moment $\vec{p} = (\overline{A}_{11}, 0, 0)$ located at $\bar{z}$ on the z axis. They are given by

$$\beta_{11} = \frac{\bar{z}\overline{A}_{11}}{2},\ \beta_{21} = \frac{2\bar{z}\beta_{11}}{3} \tag{3}$$

with the solutions

$$\bar{z} = \frac{3\beta_{21}}{2\beta_{11}},\ \overline{A}_{11} = \frac{4\beta_{11}^2}{3\beta_{21}}, \tag{4}$$

where β_{11} and β_{21} are the magnetic dipole and quadrupole moments, respectively.

3 Results

3.1 Magnetic field

For the magnetic field calculation on the basis of Eq. 1 it is necessary to know the magnetic multipole moments which are represented by Eq. 2.

In the following for both line source types the resulting magnetic moments are presented up to the octupole term. The geometrical center of the sources is assumed to be located at z_0 on the z axis outside the center of the conducting sphere.

First the line source LS1 is considered. In the order $n = 1$ Eq. 2 gives the magnetic dipole moment

$$\beta_{11} = \frac{z_0 A_{11}}{2}, \tag{5}$$

where $A_{11} = 2aI$ is the electric dipole moment of the line current I of extent $2a$. The next contribution which is obtained for $n = 2$ is the magnetic quadrupole moment

$$\beta_{21} = \frac{z_0^2 A_{11}}{3}. \tag{6}$$

For $n = 3$ the resulting magnetic octupole moments are given by

$$\beta_{31} = \frac{z_0(-a^2 + 12z_0^2)A_{11}}{48} \tag{7}$$

and

$$\beta_{33} = \frac{z_0 a^2 A_{11}}{96}. \tag{8}$$

In the case of the line source LS2 the magnetic dipole moment is

$$\beta_{11} = \frac{z_0 A_{11}}{2}, \tag{9}$$

where $A_{11} = 2qb$ is the electric dipole moment of the line source of constant current dipole density q and extent $2b$. The magnetic quadrupole contribution is given by

$$\beta_{21} = \frac{(-b^2 + 3z_0^2)A_{11}}{9}. \tag{10}$$

The magnetic octupole moments are found to be

$$\beta_{31} = \frac{z_0(-11b^2 + 12z_0^2)A_{11}}{48} \tag{11}$$

and

$$\beta_{33} = -\frac{z_0 b^2 A_{11}}{96}. \tag{12}$$

For both line source types the magnetic multipole moments up to the octupole contain all information to determine the location of the geometrical center and the extent of the sources. In the limits $a \to 0$ and $b \to 0$, respectively, these multipole moments reduce to the magnetic multipole moments for a single current dipole of the electric dipole moment A_{11} located at z_0 on the z axis [11].

The magnetic field calculated by means of the above magnetic multipole moments has for both line source types the form

$$\vec{B}(\vec{r}) = \vec{B}_{n=1}(\vec{r}) + \vec{B}_{n=2}(\vec{r}) + \vec{B}_{n=3}(\vec{r}) + \cdots. \tag{13}$$

Using Cartesian coordinates to present the contributions to the magnetic field the magnetic dipole term is given by

$$\vec{B}_{n=1}(\vec{r}) = \frac{\mu_0}{4\pi r^5}\beta_{11}(3y\vec{r} - r^2\vec{e}_y), \tag{14}$$

where $\vec{e}_x, \vec{e}_y, \vec{e}_z$ are the unit vectors. The magnetic quadrupole term is found to be

$$\vec{B}_{n=2}(\vec{r}) = \frac{3\mu_0}{4\pi r^7}\beta_{21}(5yz\vec{r} - zr^2\vec{e}_y - yr^2\vec{e}_z)\,. \quad (15)$$

The contribution of the magnetic octupole can be shown to be

$$\begin{aligned}\vec{B}_{n=3}(\vec{r}) = \frac{3\mu_0}{8\pi r^9}\{&\beta_{31}[(7z^2 - r^2)5y\vec{r} \\ &+ ((r^2 - 5z^2)\vec{e}_y - 10yz\vec{e}_z)r^2] \\ &+ 10\beta_{33}[(3x^2 - y^2)7y\vec{r} \\ &- (6xy\vec{e}_x + 3(x^2 - y^2)\vec{e}_y)r^2]\}.\end{aligned} \quad (16)$$

Now it is assumed that the geometrical center of the line sources is located at the center of the conducting sphere, i. e., $z_0 = 0$. For the line source LS1 the magnetic multipole moments up to the octupole presented above and all higher-order magnetic moments vanish. Therefore, for the line current LS1 the magnetic field $\vec{B}$ is equal to zero. In contrast, for the line source LS2 the magnetic field does not vanish. In this case the magnetic quadrupole term is found to be the lowest-order contribution to the magnetic field.

3.2 Current dipole inverse solution

The location and the moment of the ECD for the magnetic field of the line sources located near the center of the conducting sphere are obtained from Eq. 4 and the expressions for β_{11} and β_{21} presented above.
In the case of the line source LS1 the ECD result is given by

$$\bar{z} = z_0,\ \bar{A}_{11} = A_{11}\,. \quad (17)$$

Therefore, for the line current the ECD represents the location of the geometrical center on the *z* axis and the electric dipole moment.
For the line source LS2 the ECD solution is

$$\bar{z} = kz_0,\ \bar{A}_{11} = \frac{A_{11}}{k} \quad (18)$$

with

$$k = 1 - \frac{b^2}{3z_0^2}\,. \quad (19)$$

This result describes the effect of the source extent on the location and the moment of the ECD. In contrast to the line current LS1, for the source LS2 the inverse solution shows systematic deviations from the actual source parameters.

4 Discussion

The present study provides insight into the magnetic field from extended sources at deep locations in a spherical head model. The analytical results obtained by means of multipole expansion written in terms of magnetic multipole moments at the center of the conducting sphere allow a detailed analysis of the effects of the sources.
First the source center is chosen to be located at z_0 on the *z* axis outside the center of the conducting sphere. The results show that for the line current LS1 the first two multipole terms up to the magnetic quadrupole do not depend on the source extent. In contrast, the magnetic quadrupole term of the line source LS2 depends on the source extent. For both line source types the octupole contributions are influenced by the source extent. Especially, the β_{33}-term is found to be caused by the extent of the line sources. This term vanishes in both cases with vanishing source extent.
Now the source center is taken at the center of the conducting sphere, i. e., $z_0 = 0$. For the line source LS1 the magnetic field vanishes, because all magnetic multipole moments are equal to zero. However, for the line source LS2 the magnetic induction is given by

$$\vec{B}(\vec{r}) = \frac{3\mu_0}{4\pi r^7}\beta_{21}(5yz\vec{r} - zr^2\vec{e}_y - yr^2\vec{e}_z) + \cdots \quad (20)$$

with the magnetic quadrupole moment

$$\beta_{21} = -\frac{b^2 A_{11}}{9}\,. \quad (21)$$

Thus, the magnetic field does not vanish for the line source LS2 with its geometrical center at the center of the conducting sphere. This significant result is valid for all source types for which the multipole source description gives nonzero magnetic multipole moments [12].
Furthermore, in this study the multipole expansion of the magnetic field provides the basis for inverse calculations. In analogy to the electric case [9, 10] the magnetic dipole and quadrupole moments are used to determine the ECD for deep locations of the actual sources near the center of the conducting sphere. The magnetic octupole contribution and all higher-order terms are neglected. In the case of the line source LS1 the ECD describes the location of the center and the electric dipole moment of the actual source. The reason is that the first two multipole terms do not depend on the source extent. For the line source LS2 the ECD result shows

systematic deviations from the source parameters z_0 and A_{11}, because the magnetic quadrupole moment is affected by the extent of the actual source.

For the line source LS2 the location of the source center z_0 on the z axis outside the center of the conducting sphere and the electric dipole moment A_{11} can be determined by taking into account the next higher-order term of the multipole expansion, i. e., the magnetic octupole. The results for the magnetic multipole moments up to the octupole contributions presented above can be used to derive the equation

$$2\beta_{11}z_0^2 - 3\beta_{21}z_0 + 32\beta_{33} = 0 . \qquad (22)$$

The location of the source center z_0 on the z axis is given by the solution

$$z_0 = \frac{3\beta_{21}}{4\beta_{11}} + \sqrt{\frac{9\beta_{21}^2}{16\beta_{11}^2} - \frac{16\beta_{33}}{\beta_{11}}} . \qquad (23)$$

This result can be used to determine the electric dipole moment A_{11} of the line source by means of the relation

$$A_{11} = \frac{2\beta_{11}}{z_0} . \qquad (24)$$

In addition, the extent of both line sources can be obtained by taking into account the magnetic octupole term.

For the determination of the line source parameters it is necessary to know the corresponding magnetic multipole moments which can be calculated from the magnetic field outside the conducting sphere by using a surface integration method. Considering the radial component of the magnetic induction the magnetic multipole moments can be obtained by means of the equation [7]

$$\alpha_{nm} + i\beta_{nm} = \frac{2n+1}{n+1}\frac{g_{nm}}{\mu_0}\int_S r^n Y_n^m(\vartheta,\varphi)\vec{B}(\vec{r})d\vec{S} , \qquad (25)$$

where S is a spherical surface enclosing the conducting sphere.

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