Illustration of three different approaches to estimate measurement uncertainties

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https://github.com/dhueser/MDA-Vorlesung-iprom-tu-bs

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Content

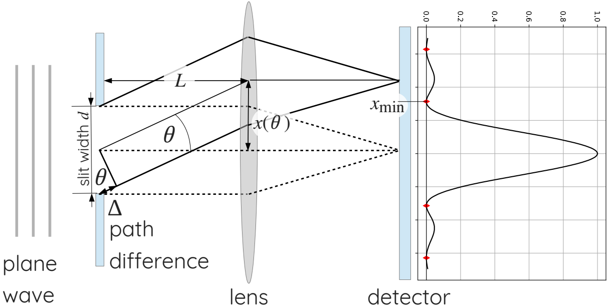
1. JCGM 100 – GUM: Uncertainty propagation of explicite, univariate and linearizable models

2. JCGM 101 – GUM-S1: Monte Carlo method - simulation of very large samples of direct measurands histogramming the indirect measurand Y_ν using the model $f(Y_\nu, (x_{1,\nu_1}, \ldots, x_{N,\nu_N})) = 0$

3. JCGM 102 – GUM-S2: Bayesian statistics

1 JCGM 100 – GUM: explicite, univariate and linearizable models

Example: Fraunhofer diffraction of single slit



direct measurands:

- distance slit – lens: L
- wavelength of light: λ
- position of x_{min} estimated from
  1. pixel positions: x
  2. gray values: $g(x)$

slit width d – coarse model:

$$d \approx \frac{\lambda L}{x_{\mathrm{min}}}$$

uncertainties are assumed to be bell like (Gaussian or t-dsitributions):

- uncertainty of L from set up: $u(L)$
- uncertainty of λ due to finite coherence: $u(\lambda)$
- uncertainty of x_{min} due to pixel noise and pixel sizes: $u(x_{\mathrm{min}})$

1 JCGM 100 – GUM: explicite, univariate and linearizable models

Example: Fraunhofer diffraction of single slit

slit width d – model equation

$$d \approx \frac{\lambda L}{x_{\mathrm{min}}}$$

combined uncertainty

$$u(d) = \sqrt{\left(c_L u(L)\right)^2 + \left(c_\lambda u(\lambda)\right)^2 + \left(c_{x_{\mathrm{min}}} u(x_{\mathrm{min}})\right)^2}$$

measurand	estimate	standard uncertainty	sensitivity	$\lvert c\,u \rvert$ in μm
L:	$\bar{L} = 4000\,\mu\mathrm{m}$	$u(L) = 20\,\mu\mathrm{m}$	$c_L = \frac{\partial}{\partial L} d\Big\vert_{\bar{L},\bar{\lambda},\bar{x}_{\mathrm{min}}} = \frac{\bar{\lambda}}{\bar{x}_{\mathrm{min}}} = 0.0014238$	0.02858
λ:	$\bar{\lambda} = 0.550\,\mu\mathrm{m}$	$u(\lambda) = 0.002\,\mu\mathrm{m}$	$c_\lambda = \frac{\partial}{\partial L} d\Big\vert_{\bar{L},\bar{\lambda},\bar{x}_{\mathrm{min}}} = \frac{\bar{L}}{\bar{x}_{\mathrm{min}}} = 10.35465$	0.02071
x_{min}:	$\bar{x}_{\mathrm{min}} = 386.3\,\mu\mathrm{m}$	$u(x_{\mathrm{min}}) = 5.9\,\mu\mathrm{m}$	$c_{x_{\mathrm{min}}} = \frac{\partial}{\partial x_{\mathrm{min}}} d\Big\vert_{\bar{L},\bar{\lambda},\bar{x}_{\mathrm{min}}} = -\frac{\bar{\lambda}\,\bar{L}}{\bar{x}_{\mathrm{min}}^2} = -0.01474257$	0.08698

$$\hookrightarrow \bar{d} \approx \frac{\bar{\lambda}\bar{L}}{\bar{x}_{\mathrm{min}}} = 5.695\,\mu\mathrm{m}$$

$$u(d) = 0.094\,\mu\mathrm{m}$$

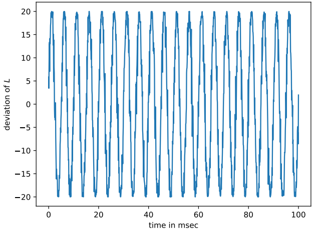
$d = (5.695 \pm 0.187)\,\mu\mathrm{m}$ for $k = 2$

2 JCGM 101 – GUM-S1: Monte Carlo method

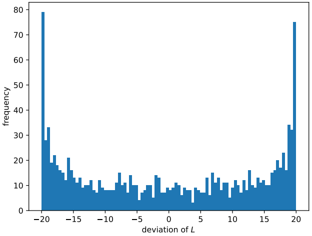
Example: Fraunhofer diffraction of single slit

may be life of an experimentalist is not so easy ...

- how to proceed if the distribution of distance L between slit and lens is not bell shaped, but U-shaped due to vibrations?



$\Rightarrow$



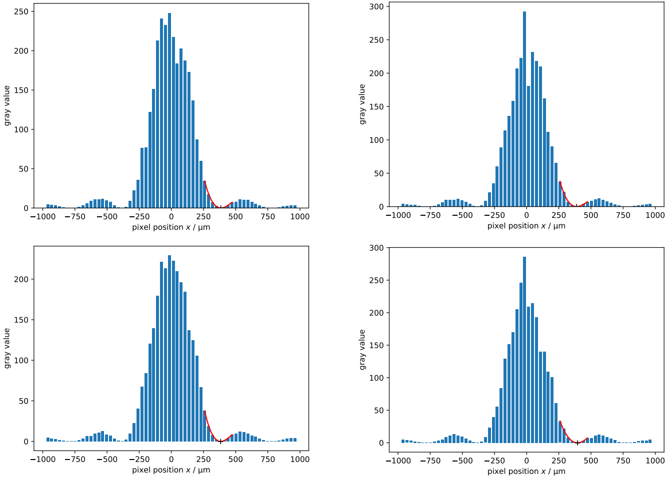
this distribution is not at all Gaussian

- which are the influences on the uncertainty of x_{min}?

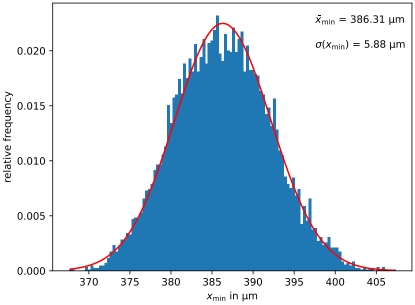
2 JCGM 101 – GUM-S1: Monte Carlo method

Example: Fraunhofer diffraction of single slit

may be life of an experimentalist is not so easy ... influences on the uncertainty of x_{min}?



- pixel noise
- pixel size precision



this distribution is not quite Gaussian

2 JCGM 101 – GUM-S1: Monte Carlo method

Example: Fraunhofer diffraction of single slit

may be life of an experimentalist is not so easy ... influences on the uncertainty of λ?

this distribution is very much a Gaussian - lucky

Gaussian spectrum of a not ideally monochromatic light source

$$S(k) = \frac{1}{2\sqrt{\pi}\,\Delta k}\, e^{-\left(\frac{k-k_0}{2\Delta k}\right)^2}$$

where k is the wave number $k = \frac{2\pi}{\lambda}$; $k_0 = \frac{2\pi}{\lambda_0}$

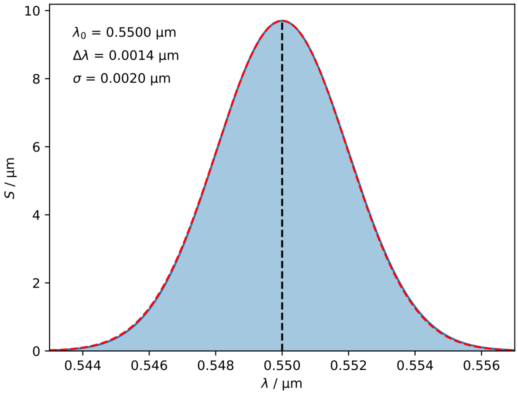
$\Delta k = \frac{2\pi}{\lambda_0}\Delta\lambda$

$$S(\lambda) = \frac{\lambda_0^2}{4\sqrt{\pi}^3\Delta\lambda}\, e^{-\left(\frac{(\lambda_0-\lambda)\lambda_0}{2\lambda\,\Delta\lambda}\right)^2}$$

$\lambda_0 = 0.5500\,\mu\mathrm{m}$ und $\Delta\lambda = 0.0014\,\mu\mathrm{m}$

$$S(\lambda) \approx a\, e^{-\frac{1}{2}\left(\frac{(\lambda_0-\lambda)}{\sigma}\right)^2}$$

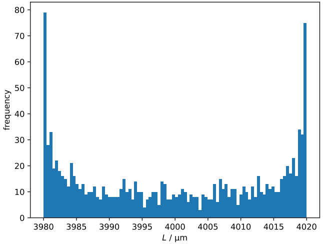
$\sigma = 0.0020\,\mu\mathrm{m}$



2 JCGM 101 – GUM-S1: Monte Carlo method

Example: Fraunhofer diffraction of single slit

use random number generator and probability densities p to generate synthetic data samples

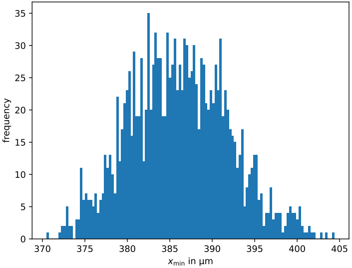


$p_L : L \mapsto p_L(L|\bar{L}, a_L)$ with

$\bar{L} = 4000\,\mu\text{m}$ and $a_L = 20\,\mu\text{m}$

$$\{L_1, \ldots, L_{i_L}, \ldots, L_{N_L}\}$$

here $N_L = 1200$

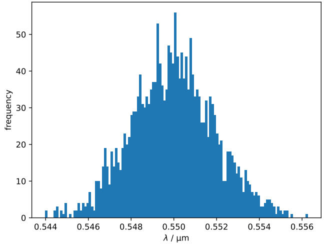


$p_{x_{\min}} : x_{\min} \mapsto p_{x_{\min}}(x_{\min}|g_{i_{x_{\min}}}(x_j), x_j)$

j: pixel index, $g_{i_{x_{\min}}}(x_j)$: gray values

$$\{x_{\min,1}, \ldots, x_{\min,i_x}, \ldots, x_{\min,N_x}\}$$

here $N_x = 1500$



$p_\lambda : \lambda \mapsto p_\lambda(\lambda|\lambda_0, \Delta\lambda)$ with

$\lambda_0 = 0.5500\,\mu\text{m}$; $\Delta\lambda = 0.0014\,\mu\text{m}$

$$\{\lambda_1, \ldots, \lambda_{i_\lambda}, \ldots, \lambda_{N_\lambda}\}$$

here $N_x = 2000$

2 JCGM 101 – GUM-S1: Monte Carlo method

Example: Fraunhofer diffraction of single slit

- generate synthetic data samples

$$\{L_1, \ldots, L_{i_L}, \ldots, L_{N_L}\} \quad \text{and} \quad \{x_{\min,1}, \ldots, x_{\min,i_x}, \ldots, x_{\min,N_x}\} \quad \text{and} \quad \{\lambda_1, \ldots, \lambda_{i_\lambda}, \ldots, \lambda_{N_\lambda}\}$$

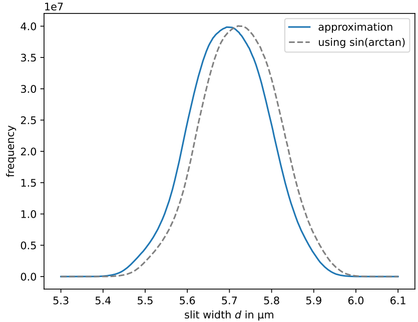
- calculate $N_L \times N_x \times N_\lambda$ slit width values using the model

$$d_{i_L,i_x,i_\lambda} = \frac{\lambda_{i_\lambda} L_{i_L}}{x_{\min,i_x}} \quad \text{resp.} \quad d_{i_L,i_x,i_\lambda} = \frac{\lambda_{i_\lambda}}{\sin\left(\arctan\left(\frac{x_{\min,i_x}}{L_{i_L}}\right)\right)}$$

histogramm $h : d_k \mapsto h(d_k)$ where d_k are the bins $k = 1, \ldots, N_{\text{bin}}$
here $N_{\text{bin}} = 300$

$$\bar{d} = \frac{1}{N_{\text{tot}}} \sum_{k=1}^{N_{\text{bin}}} d_k h(d_k) \quad \text{for} \quad N_{\text{tot}} = \sum_{k=1}^{N_{\text{bin}}} h(d_k)$$

and $$\sigma(d) = \sqrt{\frac{1}{N_{\text{tot}} - 1} \sum_{k=1}^{N_{\text{bin}}} \left(d_k - \bar{d}\right)^2 h(d_k)}$$



2 JCGM 101 – GUM-S1: Monte Carlo method

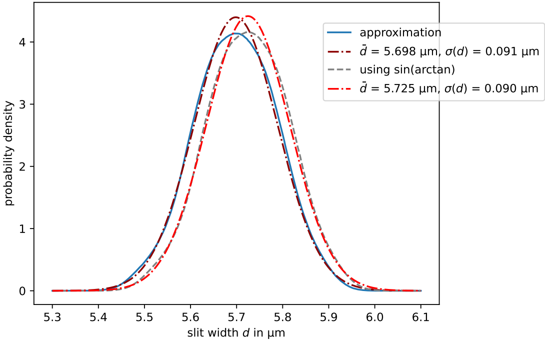
Example: Fraunhofer diffraction of single slit

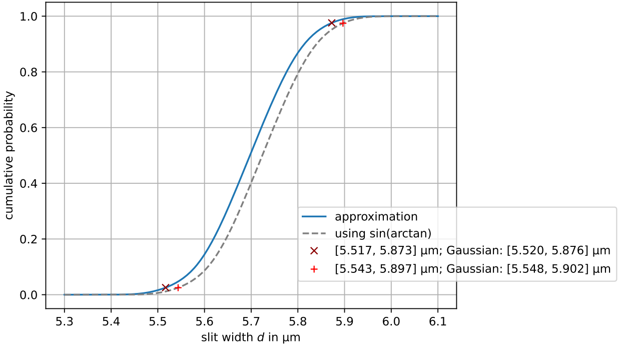
probability density of indirect measurand slit width d: $p_d : d \mapsto p_d(d)$

$$p_d(d) \;\propto\; \int \mathrm{d}L \int \mathrm{d}\lambda \int \mathrm{d}x_{\min}\; \delta(d - f(L, \lambda, x_{\min}))\; p_L(L|\bar{L}, a_L)\; p_{x_{\min}}(x_{\min}|g_{i_{x_{\min}}}(x_j), x_j)\; p_\lambda(\lambda|\lambda_0, \Delta\lambda)$$

measurement result:

- estimate of the measurand d: $\bar{d} = \int d\, p(d)\, \mathrm{d}d$
- and its uncertainty interval for 95% confidence level





3 JCGM 102 – GUM-S2: Bayesian statistics

- probability density of indirect measurand slit width d: $p_d : d \mapsto p_d(d)$

$$p_d(d) \propto \int \mathrm{d}L \int \mathrm{d}\lambda \int \mathrm{d}x_{\min}\, \delta(d - f(L, \lambda, x_{\min}))\, p_L(L|\bar{L}, a_L)\, p_{x_{\min}}(x_{\min}|g_{i_{x_{\min}}}(x_j), x_j)\, p_\lambda(\lambda|\lambda_0, \Delta\lambda)$$

- the Dirac distribution $\delta(d - f(L, \lambda, x_{\min}))$ means that for each tuple $(L_{i_L}, \lambda_{i_\lambda}, x_{\min, i_x})$ the model function $f(L, \lambda, x_{\min})$
  - is directly used
  - no uncertainty is assigned to the model itself
  - all uncertainty has its origin in the uncertainties of the direct measurands L, λ, and $x_{\min}$
- using a model means incorporating à priori knowledge, therefore the peak shaped (infinitely thin) distribution $\delta(d - f(L, \lambda, x_{\min}))$ is a prior probability denstity distribution
- in Bayesian statistics, the model prior is a probability denstity distribution of finite width: $p_f : (d, L, \lambda, x_{\min}) \mapsto p_f(d, L, \lambda, x_{\min})$
- furthermore Bayesian statistics allows for incorporating à priori knowledge of the indirect measurand: $p_d : d \mapsto p_d(d|\bar{d}_0, \sigma_{d,0})$ respectively $p_d(d|\bar{d}_0) p_\sigma(\sigma(d)|\sigma_{d,0})$

3 JCGM 102 – GUM-S2: Bayesian statistics

- probability density of indirect measurand slit width d: $p_d : d \mapsto p_d(d)$ for sharp model prior

$$p_d(d) \;\propto\; \int \mathrm{d}L \int \mathrm{d}\lambda \int \mathrm{d}x_{\min}\;\delta(d - f(L,\lambda,x_{\min}))\;p_L(L|\bar{L},a_L)\;p_{x_{\min}}(x_{\min}|g_{i_{x_{\min}}}(x_j),x_j)\;p_\lambda(\lambda|\lambda_0,\Delta\lambda)$$

- using a model means incorporating à priori knowledge, therefore the peak shaped (infinitely thin) distribution $\delta(d - f(L,\lambda,x_{\min}))$ is a prior probability denstity distribution

- in Bayesian statistics, the model prior is a probability denstity distribution of finite width: $p_f : (d,L,\lambda,x_{\min}) \mapsto p_f(d,L,\lambda,x_{\min})$

- furthermore Bayesian statistics allows for incorporating à priori knowledge of the indirect measurand: $p_d : d \mapsto p_d(d|\bar{d}_0,\sigma_{d,0})$ respectively $p_d(d|\bar{d}_0)\;p_\sigma(\sigma(d)|\sigma_{d,0})$

- probability density of indirect measurand slit width d: $p_d : d \mapsto p_d(d)$ for model prior $p_f(d,L,\lambda,x_{\min})$ and prior $p_d(d|\bar{d}_0)\;p_\sigma(\sigma(d)|\sigma_{d,0})$

$$p_d(d) \;\propto \int \mathrm{d}L \int \mathrm{d}\lambda \int \mathrm{d}x_{\min}$$

$$p_f(d,L,\lambda,x_{\min})\;p_d(d|\bar{d}_0)\;p_\sigma(\sigma(d)|\sigma_{d,0})\;p_\lambda(\lambda|\lambda_0,\Delta\lambda)\;p_L(L|\{\ldots,L_k,\ldots\})\;p_{x_{\min}}(x_{\min}|\{\ldots,g_i(\{\ldots,x_j,\ldots\}),\ldots\})$$

L_k and g_i: currently measured; $\lambda_0,\Delta\lambda$: obtained previously when initially setting up the experiment

3 JCGM 102 – GUM-S2: Bayesian statistics

- since $\lambda_0, \Delta\lambda$ were obtained previously when initially setting up the instrument, they are also priors
- L_k and g_i are currently measured observations
- their probability density distributions are referred to as likelihoods

$$p_d(d|...) \; \propto \int \mathrm{d}L \int \mathrm{d}\lambda \int \mathrm{d}x_{\mathrm{min}}$$

$$\underbrace{p_f(d, L, \lambda, x_{\mathrm{min}})\; p_d(d|\bar{d}_0)\; p_\sigma(\sigma(d)|\sigma_{d,0})\; p_\lambda(\lambda|\lambda_0, \Delta\lambda)}_{\text{prior distributions}} \; \underbrace{p_L(L|\{\dots, L_k, \dots\})\; p_{x_{\mathrm{min}}}(x_{\mathrm{min}}|\{\dots, g_i(\{\dots, x_j, \dots\}), \dots\})}_{\text{likelihoods}}$$

- the resultant probability density distribution $p_d(d|...)$ is called posterior
- now, consider a less complex example with two direct measurands, one currently measured, the other as à priori knowledge

3 JCGM 102 – GUM-S2: Bayesian statistics

Example: measurand with calibration factor

- indirect measurand: some physical quantity Y with some physical unit, denote it with the German word for unit: "Einheit" E
- direct measurand X_M: raw signal as voltage U, physical unit Volt V
- direct measurand X_K: the calibration factor K with unit $\frac{\mathrm{E}}{\mathrm{V}}$
- model

$$Y = f(X_\mathrm{K}, X_\mathrm{M}) = X_\mathrm{K}\, X_\mathrm{M}$$

3 JCGM 102 – GUM-S2: Bayesian statistics

Example: measurand with calibration factor

- Sample of observations of raw voltage signal with sample size $J_\mathrm{M} = 9$:

X_M/V	479.58	526.47	516.77	522.01	506.61	497.99	481.71	484.90	491.41

- calibration factor X_K

$$X_\mathrm{K} = (K_0 \pm U_\mathrm{K}) \frac{\mathrm{E}}{\mathrm{V}} = (0.0925 \pm 0.0180) \frac{\mathrm{E}}{\mathrm{V}} \quad \text{mit} \quad k = 2 \quad \text{und} \quad \nu_\mathrm{K} = 45$$

$k = 2$ rounded value of t quantile ($k = 2.0141$) for $95\,\%$ confidence level with $\nu_\mathrm{K} = 45$ degrees of freedom mean and empirical standard deviation from sample of direct measurand X_M:

$$\bar{x}_{\mathrm{M}} = \frac{1}{9}\sum_{j=1}^{9} X_{\mathrm{M},j} = 500.83\,\mathrm{V}$$

$$s_{\mathrm{M}} = \sqrt{\frac{1}{\nu_{\mathrm{M}}}\sum_{j=1}^{9}(X_{\mathrm{M},j} - \bar{x}_{\mathrm{M}})^2} = 17.8927\,\mathrm{V} \approx 17.89\,\mathrm{V}$$

with $\nu_{\mathrm{M}} = 8$ degrees of freedom

3 JCGM 102 – GUM-S2: Bayesian statistics

Example: measurand with calibration factor

- $s_{\mathrm{KM}} = K_0\, s_{\mathrm{M}} = 0.0925\,\frac{\mathrm{E}}{\mathrm{V}} \cdot 17.8927\,\mathrm{V} = 1.655\,\mathrm{E}$
- $s_{\mathrm{K}} = \frac{U_{\mathrm{K}}}{2} = 0.0090$
- $\sigma_{\mathrm{f}} = 0.003 \cdot s_{\mathrm{KM}} = 0.00497$
- coverage interval for 95% confidence level:
  - t-quantile for 52 d.o.f.: $k = 2.306$
  - then coverage interval: $[36.691\,\mathrm{E}, 55.962\,\mathrm{E}]$

task: use Bayesian method to determine

- the expectation value (estimate of measurand) and
- the coverage interval 95% credibility from posterior

$$\underbrace{p(Y, X_{\mathrm{K}}, X_{\mathrm{M}} | (X_{\mathrm{M},1}, \ldots, X_{\mathrm{M},J}), K_0, s_{\mathrm{K}})}_{\text{posterior}} \propto \underbrace{e^{-\frac{1}{2}\left(\frac{Y - X_{\mathrm{K}}\,X_{\mathrm{M}}}{\sigma_{\mathrm{f}}}\right)^2}}_{\text{model prior}} \underbrace{e^{-\frac{1}{2}\left(\frac{X_{\mathrm{K}} - K_0}{s_{\mathrm{K}}}\right)^2}}_{\text{prior calib.}} \underbrace{\prod_{j=1}^{J} e^{-\frac{1}{2}\left(\frac{X_{\mathrm{M}} - X_{\mathrm{M},j}}{s_{\mathrm{M}}}\right)^2}}_{\text{likelihood}}$$

marginal distribution:

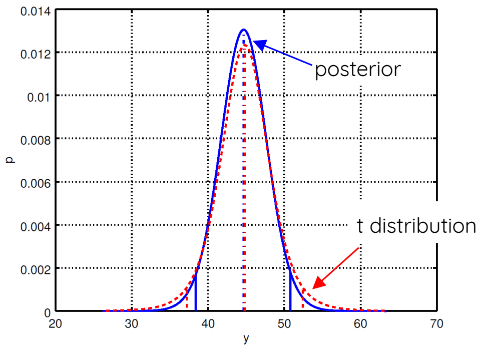
$$\underbrace{p(Y | (X_{\mathrm{M},1}, \ldots, X_{\mathrm{M},J}), K_0, s_{\mathrm{K}})}_{\text{posterior}} \propto \int \mathrm{d}X_{\mathrm{M}} \int \mathrm{d}X_{\mathrm{K}}\ \underbrace{e^{-\frac{1}{2}\left(\frac{Y - X_{\mathrm{K}}\,X_{\mathrm{M}}}{\sigma_{\mathrm{f}}}\right)^2}}_{\text{model prior}} \underbrace{e^{-\frac{1}{2}\left(\frac{X_{\mathrm{K}} - K_0}{s_{\mathrm{K}}}\right)^2}}_{\text{prior calib.}} \underbrace{\prod_{j=1}^{J} e^{-\frac{1}{2}\left(\frac{X_{\mathrm{M}} - X_{\mathrm{M},j}}{s_{\mathrm{M}}}\right)^2}}_{\text{likelihood}}$$

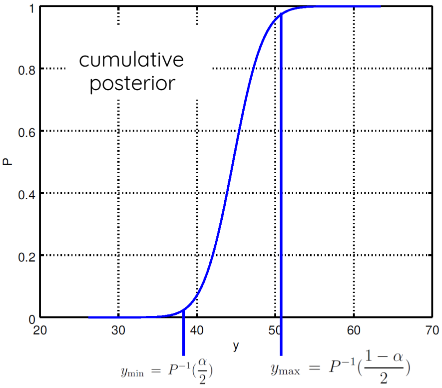
3 JCGM 102 – GUM-S2: Bayesian statistics

Example: measurand with calibration factor

task: from posterior distribution determine

- expectation value (estimate of measurand) and
- the coverage interval 95% credibility





3 JCGM 102 – GUM-S2: Bayesian statistics

Example: measurand with calibration factor

- normalize posterior
- marginal distribution

$$p(Y|(X_{\mathrm{M},1},\ldots,X_{\mathrm{M},J}),K_0,s_\mathrm{K}) = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{Y\,-\,X_\mathrm{K}\,X_\mathrm{M}}{\sigma_\mathrm{f}}\right)^2} e^{-\frac{1}{2}\left(\frac{X_\mathrm{K}-K_0}{s_\mathrm{K}}\right)^2} \prod\limits_{j=1}^{J} e^{-\frac{1}{2}\left(\frac{X_\mathrm{M}-X_{\mathrm{M},j}}{s_\mathrm{M}}\right)^2} \,\mathrm{d}\,X_\mathrm{M}\,\mathrm{d}\,X_\mathrm{K}}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{Y\,-\,X_\mathrm{K}\,X_\mathrm{M}}{\sigma_\mathrm{f}}\right)^2} e^{-\frac{1}{2}\left(\frac{X_\mathrm{K}-K_0}{s_\mathrm{K}}\right)^2} \prod\limits_{j=1}^{J} e^{-\frac{1}{2}\left(\frac{X_\mathrm{M}-X_{\mathrm{M},j}}{s_\mathrm{M}}\right)^2} \,\mathrm{d}\,X_\mathrm{M}\,\mathrm{d}\,X_\mathrm{K}\,\mathrm{d}\,Y}$$

3 JCGM 102 – GUM-S2: Bayesian statistics

Example: measurand with calibration factor

estimate $\bar{y}$ – expectation value of the posterior

$$\bar{y} = \int_{-\infty}^{\infty} Y \, p(Y|(X_{\mathrm{M},1}, \ldots, X_{\mathrm{M},J}), K_0, s_{\mathrm{K}}) \, \mathrm{d}\, Y = 46.278 \, \mathrm{E}$$

cumulative posterior to obtain coverage interval $[y_1, y_2]$

$$P(Y) = \int_{-\infty}^{Y} p(Y'|(X_{\mathrm{M},1}, \ldots, X_{\mathrm{M},J}), K_0, s_{\mathrm{K}}) \, \mathrm{d}\, Y'$$

for a probability of 95% (credibility)

$P(y_1) = 0.025 \Leftrightarrow y_1 = P^{-1}(0.025) = 36.855 \, \mathrm{E}$

$P(y_2) = 0.975 \Leftrightarrow y_2 = P^{-1}(0.975) = 56.185 \, \mathrm{E}$ thus $[\bar{y} - 9.423, \bar{y} + 9.907] \, \mathrm{E}$

3 JCGM 102 – GUM-S2: Bayesian statistics vs. JCGM 100 – GUM

Example: measurand with calibration factor

- expectation value from classical method:
  product of $\bar{x}_\mathrm{M} = 500.83\,\mathrm{V}$ and $K_0 = 0.0925\,\frac{\mathrm{E}}{\mathrm{V}}$

$$\bar{y} \;=\; K_0\,\bar{x}_\mathrm{M} \;=\; 46.327\,\mathrm{E}$$

- Bayesian result $\bar{y} \;=\; 46.278\,\mathrm{E}$
- difference $0.049\,\mathrm{E}$ – it is one promille

- uncertainty due to propagation according to GUM JCGM 100 (no correlation $\rho_{\mathrm{M,K}} = 0$)

$$u^2(Y) \;=\; \left(\frac{\partial}{\partial X_{\mathrm{M}}} X_{\mathrm{M}} X_{\mathrm{K}} \right)^2_{\bar{x}_{\mathrm{M}}, K_0} s_{\mathrm{M}}^2 \;+\; \left(\frac{\partial}{\partial X_{\mathrm{K}}} X_{\mathrm{M}} X_{\mathrm{K}} \right)^2_{\bar{x}_{\mathrm{M}}, K_0} s_{\mathrm{K}}^2$$

$$u(Y) \;=\; \sqrt{K_0^2\, s_{\mathrm{M}}^2 \;+\; \bar{x}_{\mathrm{M}}^2\, s_{\mathrm{K}}^2} \;=\; 4.802\,\mathrm{E}$$

3 JCGM 102 – GUM-S2: Bayesian statistics vs. JCGM 100 – GUM

Example: measurand with calibration factor

- number of degrees of freedom from Satterthwaite's equation

$$\frac{u(Y)^4}{\nu_y} = \frac{K_0^4\, s_{\mathrm{M}}^4}{\nu_{\mathrm{M}}} + \frac{\bar{x}_{\mathrm{M}}^4\, s_{\mathrm{K}}^4}{\nu_{\mathrm{K}}}$$

mit $\nu_{\mathrm{K}} = 45$, $\nu_{\mathrm{M}} = 8$, $K_0 = 0.0925\,\frac{\mathrm{E}}{\mathrm{V}}$, $s_{\mathrm{K}} = 0.009\,\frac{\mathrm{E}}{\mathrm{V}}$, $\bar{x}_{\mathrm{M}} = 500.83\,\mathrm{V}$, $s_{\mathrm{M}} = 17.89\,\mathrm{V}$

$$\nu_y = u(Y)^4 \left(\frac{K_0^4\, s_{\mathrm{M}}^4}{\nu_{\mathrm{M}}} + \frac{\bar{x}_{\mathrm{M}}^4\, s_{\mathrm{K}}^4}{\nu_{\mathrm{K}}} \right)^{-1} = 52.58 \approx 52$$

$$\nu_y = \left\lfloor u(Y)^4 \left(\sum_{i=1}^{N} \frac{(c_i\, s_i)^4}{\nu_i} \right)^{-1} \right\rfloor$$

3 JCGM 102 – GUM-S2: Bayesian statistics vs. JCGM 100 – GUM

Example: measurand with calibration factor compared with JCGM 100 – GUM

Confidence interval	Credible interval
$[36.691\,\mathrm{E}, 55.962\,\mathrm{E}]$	$[36.855\,\mathrm{E}, 56.185\,\mathrm{E}]$

Measurement result

$$Y = (46.327 \pm 9.635)\,\mathrm{E} \quad Y = (46.278 - 9.423 + 9.907)\,\mathrm{E}$$

3 JCGM 102 – GUM-S2: Bayesian statistics vs. JCGM 100 – GUM

The different concepts

	indirect measurands $Y_1, \dots, Y_M$	physics of measurement process →	direct measurands $X_1, \dots, X_N$
classical frequentistical	unknown constant / fixed, true Likelihood $l(P)$		random numbers (influenced by meas. process)
Bayesian	random numbers (lack of knowledge) Degree of Belief $p(Y)$, credibility	← increase of information by observations	observations are known fixed, specific values